arXiv:0909.5245v1 [math.DS] 29 Sep 2009

SOME BOUNDEDNESS RESULTS FOR SYSTEMS OF TWO RATIONAL DIFFERENCE EQUATIONS

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Abstract. We study k^{th} order systems of two rational difference equations

$$x_n = \frac{\alpha + \sum_{i=1}^{k}\beta_i x_{n-i} + \sum_{i=1}^{k}\gamma_i y_{n-i}}{A + \sum_{j=1}^{k} B_j x_{n-j} + \sum_{j=1}^{k} C_j y_{n-j}}, \quad n \in \mathbb{N},$$

$$y_n = \frac{p + \sum_{i=1}^{k}\delta_i x_{n-i} + \sum_{i=1}^{k}\epsilon_i y_{n-i}}{q + \sum_{j=1}^{k} D_j x_{n-j} + \sum_{j=1}^{k} E_j y_{n-j}}, \quad n \in \mathbb{N}.$$

In particular we assume non-negative parameters and non-negative initial conditions. We develop several approaches which allow us to extend well known boundedness results on the k^{th} order rational difference equation to the setting of systems in certain cases.

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1. Introduction

There has been recent interest in the study of systems of rational difference equations. The purpose of this article is to provide several analogues for successful techniques which were responsible for some well known boundedness results on the k^{th} order rational difference equation. The well known results we refer to are those presented in [4]. Several years following the appearance of [4] in the literature, [7] appeared in the literature. [7] primarily served to generalize some of the results presented in [4]. This was done by showing that some analogous results held for certain recursive inequalities. It turns out that this generalization is very useful when studying the boundedness character of systems of rational difference equations. Often times a complicated system will have a simpler difference inequality associated. We then use Theorem 1 of [7] to show that one of the sequences $\{x_n\}_{n=1}^{\infty}$ or $\{y_n\}_{n=1}^{\infty}$ is bounded. In this way Theorem 1 of [7] will provide the primary mechanism used in the proof of many of the results presented below.

2. Using a Comparison

Since we are presenting many results on systems of two rational difference equations which are analogues to the results of [4] and since we make heavy use of [7], it

Date: September 9, 2009.
[0]**Keywords: difference equation, systems, boundedness character.**
AMS Subject Classification: 39A10,39A11

should come as no surprise that we make use of similar notation. So we let $I_\beta = \{i \in \{1,\dots,k\} | \beta_i > 0\}$, $I_\gamma = \{i \in \{1,\dots,k\} | \gamma_i > 0\}$, $I_\delta = \{i \in \{1,\dots,k\} | \delta_i > 0\}$, $I_\epsilon = \{i \in \{1,\dots,k\} | \epsilon_i > 0\}$, $I_B = \{j \in \{1,\dots,k\} | B_j > 0\}$, $I_C = \{j \in \{1,\dots,k\} | C_j > 0\}$, $I_D = \{j \in \{1,\dots,k\} | D_j > 0\}$, and $I_E = \{j \in \{1,\dots,k\} | E_j > 0\}$. We also define $\min_+(\cdot,\cdot)$ to be the minimum of non-zero elements. This notation will allow us to shorten some arguments which we present later. In order for $\min_+(\cdot,\cdot)$ to be well defined at least one of the elements must be non-zero. For the results in this section, we assume that there exists constants $M_1, M_2 > 0$ so that $M_1 y_n \leq x_n \leq M_2 y_n$ for all $n \in \mathbb{N}$. We use this comparison and Theorem 1 of [7] to prove that every solution is bounded for certain systems of rational difference equations.

We begin with two theorems which generalize the standard iteration result to systems of two rational difference equations where there exists constants $M_1, M_2 > 0$ so that $M_1 y_n \leq x_n \leq M_2 y_n$ for all $n \in \mathbb{N}$. Eventually these two theorems are subsumed by several slightly more general theorems at the end of the article, particularly Theorem 20 and Theorem 21. In the following theorem, we use the results from [7] to iterate with respect to x_n.

Theorem 1. *Suppose that we have a k^{th} order system of two rational difference equations*

$$x_n = \frac{\alpha + \sum_{i=1}^k \beta_i x_{n-i} + \sum_{i=1}^k \gamma_i y_{n-i}}{A + \sum_{j=1}^k B_j x_{n-j} + \sum_{j=1}^k C_j y_{n-j}}, \quad n \in \mathbb{N},$$

$$y_n = \frac{p + \sum_{i=1}^k \delta_i x_{n-i} + \sum_{i=1}^k \epsilon_i y_{n-i}}{q + \sum_{j=1}^k D_j x_{n-j} + \sum_{j=1}^k E_j y_{n-j}}, \quad n \in \mathbb{N},$$

with non-negative parameters and non-negative initial conditions. Further assume that there exists constants $M_1, M_2 > 0$ so that $M_1 y_n \leq x_n \leq M_2 y_n$ for all $n \in \mathbb{N}$. Also suppose that $A > 0$ and there exists a positive integer η, such that for every sequence $\{c_m\}_{m=1}^\infty$ with $c_m \in I_\beta \cup I_\gamma$ for $m = 1, 2, \dots$ there exists positive integers, $N_1, N_2 \leq \eta$, such that $\sum_{m=N_1}^{N_2} c_m \in I_B \cup I_C$.
Then there exists $M > 0$ and $N \in \mathbb{N}$ so that given any non-negative initial conditions, we have $x_n, y_n \leq M$ for all $n > N$.

Proof. Since $M_1 y_n \leq x_n \leq M_2 y_n$ we have

$$x_n = \frac{\alpha + \sum_{i=1}^k \beta_i x_{n-i} + \sum_{i=1}^k \gamma_i y_{n-i}}{A + \sum_{j=1}^k B_j x_{n-j} + \sum_{j=1}^k C_j y_{n-j}} \leq \frac{\alpha + \sum_{i=1}^k \beta_i x_{n-i} + (\frac{1}{M_1}) \sum_{i=1}^k \gamma_i x_{n-i}}{A + \sum_{j=1}^k B_j x_{n-j} + (\frac{1}{M_2}) \sum_{j=1}^k C_j x_{n-j}}.$$

Thus we see that the sequence $\{x_n\}_{n=1}^\infty$ satisfies the above difference inequality. Moreover using Theorem 1 in [7] we see that there exists $M > 0$ and $N \in \mathbb{N}$ so that given any non-negative initial conditions, we have $x_n \leq M$ for all $n > N$. The fact that $M_1 y_n \leq x_n \leq M_2 y_n$ immediately yields the full result.

□

Now, in the following theorem, we use the results from [7] to iterate with respect to y_n.

Theorem 2. *Suppose that we have a k^{th} order system of two rational difference equations*

$$x_n = \frac{\alpha + \sum_{i=1}^k \beta_i x_{n-i} + \sum_{i=1}^k \gamma_i y_{n-i}}{A + \sum_{j=1}^k B_j x_{n-j} + \sum_{j=1}^k C_j y_{n-j}}, \quad n \in \mathbb{N},$$

$$y_n = \frac{p + \sum_{i=1}^k \delta_i x_{n-i} + \sum_{i=1}^k \epsilon_i y_{n-i}}{q + \sum_{j=1}^k D_j x_{n-j} + \sum_{j=1}^k E_j y_{n-j}}, \quad n \in \mathbb{N},$$

with non-negative parameters and non-negative initial conditions. Further assume that there exists constants $M_1, M_2 > 0$ so that $M_1 y_n \le x_n \le M_2 y_n$ for all $n \in \mathbb{N}$. Also suppose that $q > 0$ and there exists a positive integer η, such that for every sequence $\{c_m\}_{m=1}^\infty$ with $c_m \in I_\delta \cup I_\epsilon$ for $m = 1, 2, \ldots$ there exists positive integers, N_1,$N_2 \le \eta$, such that $\sum_{m=N_1}^{N_2} c_m \in I_D \cup I_E$.
Then there exists $M > 0$ and $N \in \mathbb{N}$ so that given any non-negative initial conditions, we have $x_n, y_n \le M$ for all $n > N$.

Proof. Since $M_1 y_n \le x_n \le M_2 y_n$ we have

$$y_n = \frac{p + \sum_{i=1}^k \delta_i x_{n-i} + \sum_{i=1}^k \epsilon_i y_{n-i}}{q + \sum_{j=1}^k D_j x_{n-j} + \sum_{j=1}^k E_j y_{n-j}} \le \frac{p + \sum_{i=1}^k \delta_i M_2 y_{n-i} + \sum_{i=1}^k \epsilon_i y_{n-i}}{q + \sum_{j=1}^k D_j M_1 y_{n-j} + \sum_{j=1}^k E_j y_{n-j}}.$$

Thus we see that the sequence $\{y_n\}_{n=1}^\infty$ satisfies the above difference inequality. Using Theorem 1 in [7] we see that there exists $M > 0$ and $N \in \mathbb{N}$ so that given any non-negative initial conditions, we have $y_n \le M$ for all $n > N$. The fact that $M_1 y_n \le x_n \le M_2 y_n$ yields the full result.

□

Theorem 3. *Suppose that we have a k^{th} order system of two rational difference equations*

$$x_n = \frac{\alpha + \sum_{i=1}^k \beta_i x_{n-i} + \sum_{i=1}^k \gamma_i y_{n-i}}{A + \sum_{j=1}^k B_j x_{n-j} + \sum_{j=1}^k C_j y_{n-j}}, \quad n \in \mathbb{N},$$

$$y_n = \frac{p + \sum_{i=1}^k \delta_i x_{n-i} + \sum_{i=1}^k \epsilon_i y_{n-i}}{q + \sum_{j=1}^k D_j x_{n-j} + \sum_{j=1}^k E_j y_{n-j}}, \quad n \in \mathbb{N},$$

with non-negative parameters and non-negative initial conditions. Further assume that there exists constants $M_1, M_2 > 0$ so that $M_1 y_n \le x_n \le M_2 y_n$ for all $n \in \mathbb{N}$. Also suppose that $A = 0$ and one of the following holds

(i) $I_B \cup I_C \subset I_\beta \cup I_\gamma$ *and* $I_B \ne \emptyset$
(ii) $q = 0$, $I_D \cup I_E \subset I_\delta \cup I_\epsilon$, *and* $I_C \ne \emptyset$
(iii) $p, q > 0$, $I_D \cup I_E \subset I_\delta \cup I_\epsilon$, *and* $I_C \ne \emptyset$

and there exists a positive integer η, such that for every sequence $\{c_m\}_{m=1}^\infty$ with $c_m \in I_\beta \cup I_\gamma$ for $m = 1, 2, \ldots$ there exists positive integers, N_1,$N_2 \le \eta$, such that $\sum_{m=N_1}^{N_2} c_m \in I_B \cup I_C$.
Then there exists $M > 0$ and $N \in \mathbb{N}$ so that given any non-negative initial conditions, we have $x_n, y_n \le M$ for all $n > N$.

Proof. Suppose that $I_B \cup I_C \subset I_\beta \cup I_\gamma$ and $I_B \neq \emptyset$ then we have

$$x_n = \frac{\alpha + \sum_{i=1}^k \beta_i x_{n-i} + \sum_{i=1}^k \gamma_i y_{n-i}}{\sum_{j=1}^k B_j x_{n-j} + \sum_{j=1}^k C_j y_{n-j}} \geq \frac{\sum_{i\in I_\beta} \beta_i M_1 y_{n-i} + \sum_{i\in I_\gamma} \gamma_i y_{n-i}}{\sum_{j\in I_\beta \cup I_\gamma} B_j M_2 y_{n-j} + C_j y_{n-j}}$$
$$\geq \frac{\min(\min_{i\in I_\beta}(\beta_i M_1), \min_{i\in I_\gamma}(\gamma_i))}{\max_{j\in I_\beta\cup I_\gamma}(B_j M_2 + C_j)}.$$

So in this case $\{x_n\}$ is bounded below by a constant. Now suppose that $q = 0$, $I_D \cup I_E \subset I_\delta \cup I_\epsilon$, and $I_C \neq \emptyset$ then we have

$$y_n = \frac{p + \sum_{i=1}^k \delta_i x_{n-i} + \sum_{i=1}^k \epsilon_i y_{n-i}}{\sum_{j=1}^k D_j x_{n-j} + \sum_{j=1}^k E_j y_{n-j}} \geq \frac{\sum_{i\in I_\delta} \delta_i M_1 y_{n-i} + \sum_{i\in I_\epsilon} \epsilon_i y_{n-i}}{\sum_{j\in I_\delta\cup I_\epsilon} D_j M_2 y_{n-j} + E_j y_{n-j}}$$
$$\geq \frac{\min(\min_{i\in I_\delta}(\delta_i M_1), \min_{i\in I_\epsilon}(\epsilon_i))}{\max_{j\in I_\delta\cup I_\epsilon}(D_j M_2 + E_j)}.$$

So in this case $\{y_n\}$ is bounded below by a constant. Now suppose that $p, q > 0$, $I_D \cup I_E \subset I_\delta \cup I_\epsilon$, and $I_C \neq \emptyset$ then we have

$$y_n = \frac{p + \sum_{i=1}^k \delta_i x_{n-i} + \sum_{i=1}^k \epsilon_i y_{n-i}}{q + \sum_{j=1}^k D_j x_{n-j} + \sum_{j=1}^k E_j y_{n-j}} \geq \frac{p + \sum_{i\in I_\delta} \delta_i M_1 y_{n-i} + \sum_{i\in I_\epsilon} \epsilon_i y_{n-i}}{q + \sum_{j\in I_\delta\cup I_\epsilon} D_j M_2 y_{n-j} + E_j y_{n-j}}$$
$$\geq \frac{\min(p, \min_{i\in I_\delta}(\delta_i M_1), \min_{i\in I_\epsilon}(\epsilon_i))}{\max_{j\in I_\delta\cup I_\epsilon}(q + D_j M_2 + E_j)}.$$

So in this case $\{y_n\}$ is bounded below by a constant. Now we have shown that there exists $A_2 > 0$, so that $\{x_n\}$ is bounded below by A_2 and $I_B \neq \emptyset$ or $\{y_n\}$ is bounded below by A_2 and $I_C \neq \emptyset$. We use this fact to show the following.

$$x_n = \frac{\alpha + \sum_{i=1}^k \beta_i x_{n-i} + \sum_{i=1}^k \gamma_i y_{n-i}}{\sum_{j=1}^k B_j x_{n-j} + \sum_{j=1}^k C_j y_{n-j}}$$
$$\leq \frac{\alpha + \sum_{i=1}^k \beta_i x_{n-i} + (\frac{1}{M_1})\sum_{i=1}^k \gamma_i x_{n-i}}{\min_+(\sum_{j=1}^k B_j, \sum_{j=1}^k C_j)\frac{A_2}{2} + \frac{1}{2}\sum_{j=1}^k B_j x_{n-j} + \frac{1}{2}(\frac{1}{M_2})\sum_{j=1}^k C_j x_{n-j}}.$$

Thus we see that the sequence $\{x_n\}_{n=1}^\infty$ satisfies the above difference inequality. Moreover using Theorem 1 in [7] we see that there exists $M > 0$ and $N \in \mathbb{N}$ so that given any non-negative initial conditions, we have $x_n \leq M$ for all $n > N$. The fact that $M_1 y_n \leq x_n \leq M_2 y_n$ immediately yields the full result.

□

Now, in the following theorem, we use the results from [7] to iterate with respect to y_n.

Theorem 4. *Suppose that we have a k^{th} order system of two rational difference equations*

$$x_n = \frac{\alpha + \sum_{i=1}^k \beta_i x_{n-i} + \sum_{i=1}^k \gamma_i y_{n-i}}{A + \sum_{j=1}^k B_j x_{n-j} + \sum_{j=1}^k C_j y_{n-j}}, \quad n \in \mathbb{N},$$

$$y_n = \frac{p + \sum_{i=1}^{k} \delta_i x_{n-i} + \sum_{i=1}^{k} \epsilon_i y_{n-i}}{q + \sum_{j=1}^{k} D_j x_{n-j} + \sum_{j=1}^{k} E_j y_{n-j}}, \quad n \in \mathbb{N},$$

with non-negative parameters and non-negative initial conditions. Further assume that there exists constants $M_1, M_2 > 0$ so that $M_1 y_n \leq x_n \leq M_2 y_n$ for all $n \in \mathbb{N}$. Also suppose that $q = 0$ and one of the following holds

(i) $A = 0$, $I_B \cup I_C \subset I_\beta \cup I_\gamma$, *and* $I_D \neq \emptyset$

(ii) $\alpha, A > 0$, $I_B \cup I_C \subset I_\beta \cup I_\gamma$, *and* $I_D \neq \emptyset$

(iii) $I_D \cup I_E \subset I_\delta \cup I_\epsilon$ *and* $I_E \neq \emptyset$

and there exists a positive integer η, such that for every sequence $\{c_m\}_{m=1}^{\infty}$ with $c_m \in I_\delta \cup I_\epsilon$ for $m = 1, 2, \ldots$ there exists positive integers, N_1,$N_2 \leq \eta$, such that $\sum_{m=N_1}^{N_2} c_m \in I_D \cup I_E$.

Then there exists $M > 0$ and $N \in \mathbb{N}$ so that given any non-negative initial conditions, we have $x_n, y_n \leq M$ for all $n > N$.

Proof. Suppose that $A = 0$, $I_B \cup I_C \subset I_\beta \cup I_\gamma$, and $I_D \neq \emptyset$ then we have

$$x_n = \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{n-i} + \sum_{i=1}^{k} \gamma_i y_{n-i}}{\sum_{j=1}^{k} B_j x_{n-j} + \sum_{j=1}^{k} C_j y_{n-j}} \geq \frac{\sum_{i \in I_\beta} \beta_i M_1 y_{n-i} + \sum_{i \in I_\gamma} \gamma_i y_{n-i}}{\sum_{j \in I_\beta \cup I_\gamma} B_j M_2 y_{n-j} + C_j y_{n-j}}$$

$$\geq \frac{\min(\min_{i \in I_\beta}(\beta_i M_1), \min_{i \in I_\gamma}(\gamma_i))}{\max_{j \in I_\beta \cup I_\gamma}(B_j M_2 + C_j)}.$$

So in this case $\{x_n\}$ is bounded below by a constant. Now suppose that $A, \alpha > 0$, $I_B \cup I_C \subset I_\beta \cup I_\gamma$, and $I_D \neq \emptyset$ then we have

$$x_n = \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{n-i} + \sum_{i=1}^{k} \gamma_i y_{n-i}}{A + \sum_{j=1}^{k} B_j x_{n-j} + \sum_{j=1}^{k} C_j y_{n-j}} \geq \frac{\alpha + \sum_{i \in I_\beta} \beta_i M_1 y_{n-i} + \sum_{i \in I_\gamma} \gamma_i y_{n-i}}{A + \sum_{j \in I_\beta \cup I_\gamma} B_j M_2 y_{n-j} + C_j y_{n-j}}$$

$$\geq \frac{\min(\alpha, \min_{i \in I_\beta}(\beta_i M_1), \min_{i \in I_\gamma}(\gamma_i))}{\max_{j \in I_\beta \cup I_\gamma}(A + B_j M_2 + C_j)}.$$

So in this case $\{x_n\}$ is bounded below by a constant. Now suppose that $I_D \cup I_E \subset I_\delta \cup I_\epsilon$, and $I_E \neq \emptyset$ then we have

$$y_n = \frac{p + \sum_{i=1}^{k} \delta_i x_{n-i} + \sum_{i=1}^{k} \epsilon_i y_{n-i}}{\sum_{j=1}^{k} D_j x_{n-j} + \sum_{j=1}^{k} E_j y_{n-j}} \geq \frac{\sum_{i \in I_\delta} \delta_i M_1 y_{n-i} + \sum_{i \in I_\epsilon} \epsilon_i y_{n-i}}{\sum_{j \in I_\delta \cup I_\epsilon} D_j M_2 y_{n-j} + E_j y_{n-j}}$$

$$\geq \frac{\min(\min_{i \in I_\delta}(\delta_i M_1), \min_{i \in I_\epsilon}(\epsilon_i))}{\max_{j \in I_\delta \cup I_\epsilon}(D_j M_2 + E_j)}.$$

So in this case $\{y_n\}$ is bounded below by a constant. Now we have shown that there exists $q_2 > 0$, so that $\{x_n\}$ is bounded below by q_2 and $I_D \neq \emptyset$ or $\{y_n\}$ is bounded below by q_2 and $I_E \neq \emptyset$. We use this fact to show the following.

$$y_n = \frac{p + \sum_{i=1}^{k} \delta_i x_{n-i} + \sum_{i=1}^{k} \epsilon_i y_{n-i}}{\sum_{j=1}^{k} D_j x_{n-j} + \sum_{j=1}^{k} E_j y_{n-j}}$$

$$\leq \frac{p+\sum_{i=1}^k \delta_i M_2 y_{n-i} + \sum_{i=1}^k \epsilon_i y_{n-i}}{\min_+(\sum_{j=1}^k D_j, \sum_{j=1}^k E_j)\frac{q_2}{2} + \frac{1}{2}\sum_{j=1}^k D_j M_1 y_{n-j} + \frac{1}{2}\sum_{j=1}^k E_j y_{n-j}}.$$

Thus we see that the sequence $\{y_n\}_{n=1}^\infty$ satisfies the above difference inequality. Again using Theorem 1 in [7] we see that there exists $M > 0$ and $N \in \mathbb{N}$ so that given any non-negative initial conditions, we have $y_n \leq M$ for all $n > N$. The fact that $M_1 y_n \leq x_n \leq M_2 y_n$ yields the full result.

□

For the following two theorems, no iteration is necessary. We only need to use some algebraic techniques similar to those used in [4], coupled with the condition that there exists constants $M_1, M_2 > 0$ so that $M_1 y_n \leq x_n \leq M_2 y_n$ for all $n \in \mathbb{N}$.

Theorem 5. *Suppose that we have a k^{th} order system of two rational difference equations*

$$x_n = \frac{\alpha + \sum_{i=1}^k \beta_i x_{n-i} + \sum_{i=1}^k \gamma_i y_{n-i}}{A + \sum_{j=1}^k B_j x_{n-j} + \sum_{j=1}^k C_j y_{n-j}}, \quad n \in \mathbb{N},$$

$$y_n = \frac{p + \sum_{i=1}^k \delta_i x_{n-i} + \sum_{i=1}^k \epsilon_i y_{n-i}}{q + \sum_{j=1}^k D_j x_{n-j} + \sum_{j=1}^k E_j y_{n-j}}, \quad n \in \mathbb{N},$$

with non-negative parameters and non-negative initial conditions. Further assume that there exists constants $M_1, M_2 > 0$ so that $M_1 y_n \leq x_n \leq M_2 y_n$ for all $n \in \mathbb{N}$. Also suppose that $q = 0$, $p = 0$, and $I_\delta \cup I_\epsilon \subset I_D \cup I_E$.
Then there exists $M > 0$ and $N \in \mathbb{N}$ so that given any non-negative initial conditions, we have $x_n, y_n \leq M$ for all $n > N$.

Proof. We use the fact that $M_1 y_n \leq x_n \leq M_2 y_n$ to show

$$y_n = \frac{\sum_{i=1}^k \delta_i x_{n-i} + \sum_{i=1}^k \epsilon_i y_{n-i}}{\sum_{j=1}^k D_j x_{n-j} + \sum_{j=1}^k E_j y_{n-j}} \leq \frac{\sum_{i=1}^k \delta_i M_2 y_{n-i} + \sum_{i=1}^k \epsilon_i y_{n-i}}{\sum_{j=1}^k D_j M_1 y_{n-j} + \sum_{j=1}^k E_j y_{n-j}} \leq$$

$$\frac{\sum_{i\in I_\delta} \delta_i M_2 y_{n-i} + \sum_{i\in I_\epsilon} \epsilon_i y_{n-i}}{\sum_{j\in I_\delta\cup I_\epsilon} D_j M_1 y_{n-j} + \sum_{j\in I_\delta\cup I_\epsilon} E_j y_{n-j}} = \frac{\sum_{i\in I_\delta\cup I_\epsilon}(\delta_i M_2 + \epsilon_i) y_{n-i}}{\sum_{j\in I_\delta\cup I_\epsilon}(D_j M_1 + E_j) y_{n-j}} \leq$$

$$\left(\frac{\max_{i\in I_\delta\cup I_\epsilon}(\delta_i M_2 + \epsilon_i)}{\min_{j\in I_\delta\cup I_\epsilon}(D_j M_1 + E_j)}\right)\frac{\sum_{i\in I_\delta\cup I_\epsilon} y_{n-i}}{\sum_{j\in I_\delta\cup I_\epsilon} y_{n-j}} = \frac{\max_{i\in I_\delta\cup I_\epsilon}(\delta_i M_2 + \epsilon_i)}{\min_{j\in I_\delta\cup I_\epsilon}(D_j M_1 + E_j)}.$$

□

Theorem 6. *Suppose that we have a k^{th} order system of two rational difference equations*

$$x_n = \frac{\alpha + \sum_{i=1}^k \beta_i x_{n-i} + \sum_{i=1}^k \gamma_i y_{n-i}}{A + \sum_{j=1}^k B_j x_{n-j} + \sum_{j=1}^k C_j y_{n-j}}, \quad n \in \mathbb{N},$$

$$y_n = \frac{p + \sum_{i=1}^k \delta_i x_{n-i} + \sum_{i=1}^k \epsilon_i y_{n-i}}{q + \sum_{j=1}^k D_j x_{n-j} + \sum_{j=1}^k E_j y_{n-j}}, \quad n \in \mathbb{N},$$

with non-negative parameters and non-negative initial conditions. Further assume that there exists constants $M_1, M_2 > 0$ so that $M_1 y_n \leq x_n \leq M_2 y_n$ for all $n \in \mathbb{N}$. Also

suppose that $A = 0$, $\alpha = 0$, and $I_\beta \cup I_\gamma \subset I_B \cup I_C$.
Then there exists $M > 0$ and $N \in \mathbb{N}$ so that given any non-negative initial conditions, we have $x_n, y_n \le M$ for all $n > N$.

Proof. Since $M_1 y_n \le x_n \le M_2 y_n$ we have

$$x_n = \frac{\sum_{i=1}^{k}\beta_i x_{n-i} + \sum_{i=1}^{k}\gamma_i y_{n-i}}{\sum_{j=1}^{k}B_j x_{n-j} + \sum_{j=1}^{k}C_j y_{n-j}} \le \frac{\sum_{i=1}^{k}\beta_i x_{n-i} + (\frac{1}{M_1})\sum_{i=1}^{k}\gamma_i x_{n-i}}{\sum_{j=1}^{k}B_j x_{n-j} + (\frac{1}{M_2})\sum_{j=1}^{k}C_j x_{n-j}} \le$$

$$\frac{\sum_{i\in I_\beta}\beta_i x_{n-i} + (\frac{1}{M_1})\sum_{i\in I_\gamma}\gamma_i x_{n-i}}{\sum_{j\in I_\beta\cup I_\gamma}B_j x_{n-j} + (\frac{1}{M_2})\sum_{j\in I_\beta\cup I_\gamma}C_j x_{n-j}} = \frac{\sum_{i\in I_\beta\cup I_\gamma}(\beta_i + \frac{\gamma_i}{M_1})x_{n-i}}{\sum_{j\in I_\beta\cup I_\gamma}(B_j + \frac{C_j}{M_2})x_{n-j}} \le$$

$$\left(\frac{\max_{i\in I_\beta\cup I_\gamma}(\beta_i + \frac{\gamma_i}{M_1})}{\min_{j\in I_\beta\cup I_\gamma}(B_j + \frac{C_j}{M_2})}\right)\frac{\sum_{i\in I_\beta\cup I_\gamma}x_{n-i}}{\sum_{j\in I_\beta\cup I_\gamma}x_{n-j}} = \frac{\max_{i\in I_\beta\cup I_\gamma}(\beta_i + \frac{\gamma_i}{M_1})}{\min_{j\in I_\beta\cup I_\gamma}(B_j + \frac{C_j}{M_2})}.$$

□

3. Some boundedness Results Without Comparison

In this section we use some algebraic techniques similar to those applied in [4]. Once we have obtained a difference inequality we apply Theorem 1 of [7].

Theorem 7. *Suppose that we have a k^{th} order system of two rational difference equations*

$$x_n = \frac{\alpha + \sum_{i=1}^{k}\beta_i x_{n-i} + \sum_{i=1}^{k}\gamma_i y_{n-i}}{A + \sum_{j=1}^{k}B_j x_{n-j} + \sum_{j=1}^{k}C_j y_{n-j}}, \quad n \in \mathbb{N},$$

$$y_n = \frac{p + \sum_{i=1}^{k}\delta_i x_{n-i} + \sum_{i=1}^{k}\epsilon_i y_{n-i}}{q + \sum_{j=1}^{k}D_j x_{n-j} + \sum_{j=1}^{k}E_j y_{n-j}}, \quad n \in \mathbb{N},$$

with non-negative parameters and non-negative initial conditions. Also suppose that $A > 0$, $I_\gamma \subset I_C$, and there exists a positive integer η, such that for every sequence $\{c_m\}_{m=1}^{\infty}$ with $c_m \in I_\beta$ for $m = 1, 2, \ldots$ there exists positive integers, N_1,$N_2 \le \eta$, such that $\sum_{m=N_1}^{N_2} c_m \in I_B$.
Then there exists $M > 0$ and $N \in \mathbb{N}$ so that given any non-negative initial conditions, we have $x_n \le M$ for all $n > N$.

Proof.

$$x_n = \frac{\alpha + \sum_{i=1}^{k}\beta_i x_{n-i} + \sum_{i=1}^{k}\gamma_i y_{n-i}}{A + \sum_{j=1}^{k}B_j x_{n-j} + \sum_{j=1}^{k}C_j y_{n-j}} \le$$

$$\frac{\sum_{i=1}^{k}\gamma_i y_{n-i}}{A + \sum_{j=1}^{k}B_j x_{n-j} + \sum_{j=1}^{k}C_j y_{n-j}} + \frac{\alpha + \sum_{i=1}^{k}\beta_i x_{n-i}}{A + \sum_{j=1}^{k}B_j x_{n-j} + \sum_{j=1}^{k}C_j y_{n-j}} \le$$

$$\sum_{i\in I_\gamma}\frac{\gamma_i}{C_i} + \frac{\alpha + \sum_{i=1}^{k}\beta_i x_{n-i}}{A + \sum_{j=1}^{k}B_j x_{n-j}}.$$

Thus we see that the sequence $\{x_n\}_{n=1}^{\infty}$ satisfies the above difference inequality. Moreover using Theorem 1 in [7] we see that there exists $M > 0$ and $N \in \mathbb{N}$ so that given any

non-negative initial conditions, we have $x_n \leq M$ for all $n > N$.

□

Theorem 8. *Suppose that we have a k^{th} order system of two rational difference equations*

$$x_n = \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{n-i} + \sum_{i=1}^{k} \gamma_i y_{n-i}}{A + \sum_{j=1}^{k} B_j x_{n-j} + \sum_{j=1}^{k} C_j y_{n-j}}, \quad n \in \mathbb{N},$$

$$y_n = \frac{p + \sum_{i=1}^{k} \delta_i x_{n-i} + \sum_{i=1}^{k} \epsilon_i y_{n-i}}{q + \sum_{j=1}^{k} D_j x_{n-j} + \sum_{j=1}^{k} E_j y_{n-j}}, \quad n \in \mathbb{N},$$

with non-negative parameters and non-negative initial conditions. Also suppose that $A = 0$, $\alpha = 0$, $I_\beta \subset I_B$, and $I_\gamma \subset I_C$.
Then there exists $M > 0$ and $N \in \mathbb{N}$ so that given any non-negative initial conditions, we have $x_n \leq M$ for all $n > N$.

Proof.

$$x_n = \frac{\sum_{i=1}^{k} \beta_i x_{n-i} + \sum_{i=1}^{k} \gamma_i y_{n-i}}{\sum_{j=1}^{k} B_j x_{n-j} + \sum_{j=1}^{k} C_j y_{n-j}} \leq$$

$$\left(\frac{\max_{i \in I_\beta \cup I_\gamma}(\beta_i + \gamma_i)}{\min(\min_{j \in I_\beta}(B_j), \min_{j \in I_\gamma}(C_j))} \right) \frac{\sum_{i \in I_\beta} x_{n-i} + \sum_{i \in I_\gamma} y_{n-i}}{\sum_{j \in I_\beta} x_{n-j} + \sum_{j \in I_\gamma} y_{n-j}} =$$

$$\frac{\max_{i \in I_\beta \cup I_\gamma}(\beta_i + \gamma_i)}{\min(\min_{j \in I_\beta}(B_j), \min_{j \in I_\gamma}(C_j))}.$$

□

Theorem 9. *Suppose that we have a k^{th} order system of two rational difference equations*

$$x_n = \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{n-i} + \sum_{i=1}^{k} \gamma_i y_{n-i}}{A + \sum_{j=1}^{k} B_j x_{n-j} + \sum_{j=1}^{k} C_j y_{n-j}}, \quad n \in \mathbb{N},$$

$$y_n = \frac{p + \sum_{i=1}^{k} \delta_i x_{n-i} + \sum_{i=1}^{k} \epsilon_i y_{n-i}}{q + \sum_{j=1}^{k} D_j x_{n-j} + \sum_{j=1}^{k} E_j y_{n-j}}, \quad n \in \mathbb{N},$$

with non-negative parameters and non-negative initial conditions. Also suppose that $\{y_n\}_{n=1}^{\infty}$ is bounded above and below, $I_C \neq \emptyset$, and there exists a positive integer η, such that for every sequence $\{c_m\}_{m=1}^{\infty}$ with $c_m \in I_\beta$ for $m = 1, 2, \ldots$ there exists positive integers, N_1,$N_2 \leq \eta$, such that $\sum_{m=N_1}^{N_2} c_m \in I_B$.
Then there exists $M > 0$ and $N \in \mathbb{N}$ so that given any non-negative initial conditions, we have $x_n \leq M$ for all $n > N$.

Proof. We know that $\{y_n\}_{n=1}^{\infty}$ is bounded above and below let us call those bounds $m_1 \leq y_n \leq m_2$. This means that we have

$$x_n = \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{n-i} + \sum_{i=1}^{k} \gamma_i y_{n-i}}{A + \sum_{j=1}^{k} B_j x_{n-j} + \sum_{j=1}^{k} C_j y_{n-j}} \leq \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{n-i} + \left(\sum_{i=1}^{k} \gamma_i\right) m_2}{\left(\sum_{j=1}^{k} C_j\right) m_1 + \sum_{j=1}^{k} B_j x_{n-j}}$$

Thus we see that the sequence $\{x_n\}_{n=1}^{\infty}$ satisfies the above difference inequality. Moreover using Theorem 1 in [7] we see that there exists $M > 0$ and $N \in \mathbb{N}$ so that given any non-negative initial conditions, we have $x_n \leq M$ for all $n > N$.

□

Theorem 10. *Suppose that we have a k^{th} order system of two rational difference equations*

$$x_n = \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{n-i} + \sum_{i=1}^{k} \gamma_i y_{n-i}}{A + \sum_{j=1}^{k} B_j x_{n-j} + \sum_{j=1}^{k} C_j y_{n-j}}, \quad n \in \mathbb{N},$$

$$y_n = \frac{p + \sum_{i=1}^{k} \delta_i x_{n-i} + \sum_{i=1}^{k} \epsilon_i y_{n-i}}{q + \sum_{j=1}^{k} D_j x_{n-j} + \sum_{j=1}^{k} E_j y_{n-j}}, \quad n \in \mathbb{N},$$

with non-negative parameters and non-negative initial conditions. Also suppose that $A = 0$ and one of the following holds

(i) $I_B \subset I_\beta$, $I_C \subset I_\gamma$ and $I_B \neq \emptyset$
(ii) $q = 0$, $I_D \subset I_\delta$, $I_E \subset I_\epsilon$, and $I_C \neq \emptyset$
(iii) $p, q > 0$, $I_D \subset I_\delta$, $I_E \subset I_\epsilon$, and $I_C \neq \emptyset$

$I_\gamma \subset I_C$, and there exists a positive integer η, such that for every sequence $\{c_m\}_{m=1}^{\infty}$ with $c_m \in I_\beta$ for $m = 1, 2, \ldots$ there exists positive integers, N_1,$N_2 \leq \eta$, such that $\sum_{m=N_1}^{N_2} c_m \in I_B$.
Then there exists $M > 0$ and $N \in \mathbb{N}$ so that given any non-negative initial conditions, we have $x_n \leq M$ for all $n > N$.

Proof. Suppose $I_B \subset I_\beta$, $I_C \subset I_\gamma$, and $I_B \neq \emptyset$ then we have

$$x_n = \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{n-i} + \sum_{i=1}^{k} \gamma_i y_{n-i}}{\sum_{j=1}^{k} B_j x_{n-j} + \sum_{j=1}^{k} C_j y_{n-j}} \geq \frac{\sum_{i \in I_\beta} \beta_i x_{n-i} + \sum_{i \in I_\gamma} \gamma_i y_{n-i}}{\sum_{j \in I_\beta} B_j x_{n-j} + \sum_{j \in I_\gamma} C_j y_{n-j}}$$
$$\geq \frac{\min(\min_{i \in I_\beta}(\beta_i), \min_{i \in I_\gamma}(\gamma_i))}{\max(\max_{j \in I_\beta}(B_j), \max_{j \in I_\gamma}(C_j))}.$$

So in this case $\{x_n\}$ is bounded below by a constant. Now suppose $q = 0$, $I_D \subset I_\delta$, $I_E \subset I_\epsilon$, and $I_C \neq \emptyset$ then we have

$$y_n = \frac{p + \sum_{i=1}^{k} \delta_i x_{n-i} + \sum_{i=1}^{k} \epsilon_i y_{n-i}}{\sum_{j=1}^{k} D_j x_{n-j} + \sum_{j=1}^{k} E_j y_{n-j}} \geq \frac{\sum_{i \in I_\delta} \delta_i x_{n-i} + \sum_{i \in I_\epsilon} \epsilon_i y_{n-i}}{\sum_{j \in I_\delta} D_j x_{n-j} + \sum_{j \in I_\epsilon} E_j y_{n-j}}$$
$$\geq \frac{\min(\min_{i \in I_\delta}(\delta_i), \min_{i \in I_\epsilon}(\epsilon_i))}{\max(\max_{j \in I_\delta}(D_j), \max_{j \in I_\epsilon}(E_j))}.$$

So in this case $\{y_n\}$ is bounded below by a constant. Now suppose $p, q > 0$, $I_D \subset I_\delta$, $I_E \subset I_\epsilon$, and $I_C \neq \emptyset$ then we have

$$y_n = \frac{p + \sum_{i=1}^{k} \delta_i x_{n-i} + \sum_{i=1}^{k} \epsilon_i y_{n-i}}{q + \sum_{j=1}^{k} D_j x_{n-j} + \sum_{j=1}^{k} E_j y_{n-j}} \geq \frac{p + \sum_{i \in I_\delta} \delta_i x_{n-i} + \sum_{i \in I_\epsilon} \epsilon_i y_{n-i}}{q + \sum_{j \in I_\delta} D_j x_{n-j} + \sum_{j \in I_\epsilon} E_j y_{n-j}}$$
$$\geq \frac{\min(p, \min_{i \in I_\delta}(\delta_i), \min_{i \in I_\epsilon}(\epsilon_i))}{\max(q, \max_{j \in I_\delta}(D_j), \max_{j \in I_\epsilon}(E_j))}.$$

So in this case $\{y_n\}$ is bounded below by a constant. Now we have shown that there exists $A_2 > 0$, so that $\{x_n\}$ is bounded below by A_2 and $I_B \neq \emptyset$ or $\{y_n\}$ is bounded below by A_2 and $I_C \neq \emptyset$. We use this fact to show the following.

$$x_n = \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{n-i} + \sum_{i=1}^{k} \gamma_i y_{n-i}}{\sum_{j=1}^{k} B_j x_{n-j} + \sum_{j=1}^{k} C_j y_{n-j}} \leq \frac{\sum_{i=1}^{k} \gamma_i y_{n-i}}{\sum_{j=1}^{k} B_j x_{n-j} + \sum_{j=1}^{k} C_j y_{n-j}}$$

$$+ \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{n-i}}{\min_{+}(\sum_{j=1}^{k} B_j, \sum_{j=1}^{k} C_j)\frac{A_2}{2} + \frac{1}{2}\sum_{j=1}^{k} B_j x_{n-j} + \frac{1}{2}\sum_{j=1}^{k} C_j y_{n-j}}$$

$$\leq \sum_{i \in I_\gamma} \frac{\gamma_i}{C_i} + \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{n-i}}{\min_{+}(\sum_{j=1}^{k} B_j, \sum_{j=1}^{k} C_j)\frac{A_2}{2} + \frac{1}{2}\sum_{j=1}^{k} B_j x_{n-j}}.$$

Thus we see that the sequence $\{x_n\}_{n=1}^{\infty}$ satisfies the above difference inequality. Moreover using Theorem 1 in [7] we see that there exists $M > 0$ and $N \in \mathbb{N}$ so that given any non-negative initial conditions, we have $x_n \leq M$ for all $n > N$.

□

Theorem 11. *Suppose that we have a k^{th} order system of two rational difference equations*

$$x_n = \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{n-i} + \sum_{i=1}^{k} \gamma_i y_{n-i}}{A + \sum_{j=1}^{k} B_j x_{n-j} + \sum_{j=1}^{k} C_j y_{n-j}}, \quad n \in \mathbb{N},$$

$$y_n = \frac{p + \sum_{i=1}^{k} \delta_i x_{n-i} + \sum_{i=1}^{k} \epsilon_i y_{n-i}}{q + \sum_{j=1}^{k} D_j x_{n-j} + \sum_{j=1}^{k} E_j y_{n-j}}, \quad n \in \mathbb{N},$$

with non-negative parameters and non-negative initial conditions. Also suppose that $\{y_n\}_{n=1}^{\infty}$ is bounded above, $A > 0$, and there exists a positive integer η, such that for every sequence $\{c_m\}_{m=1}^{\infty}$ with $c_m \in I_\beta$ for $m = 1, 2, \ldots$ there exists positive integers, N_1,$N_2 \leq \eta$, such that $\sum_{m=N_1}^{N_2} c_m \in I_B$.
Then there exists $M > 0$ and $N \in \mathbb{N}$ so that given any non-negative initial conditions, we have $x_n \leq M$ for all $n > N$.

Proof. We know that $\{y_n\}_{n=1}^{\infty}$ is bounded above, let us call that bound $y_n \leq m_3$.

$$x_n = \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{n-i} + \sum_{i=1}^{k} \gamma_i y_{n-i}}{A + \sum_{j=1}^{k} B_j x_{n-j} + \sum_{j=1}^{k} C_j y_{n-j}} \leq$$

$$\frac{\sum_{i=1}^{k} \gamma_i m_3}{A} + \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{n-i}}{A + \sum_{j=1}^{k} B_j x_{n-j}}.$$

Thus we see that the sequence $\{x_n\}_{n=1}^{\infty}$ satisfies the above difference inequality. Moreover using Theorem 1 in [7] we see that there exists $M > 0$ and $N \in \mathbb{N}$ so that given any non-negative initial conditions, we have $x_n \leq M$ for all $n > N$.

□

Theorem 12. *Suppose that we have a k^{th} order system of two rational difference equations*

$$x_n = \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{n-i} + \sum_{i=1}^{k} \gamma_i y_{n-i}}{A + \sum_{j=1}^{k} B_j x_{n-j} + \sum_{j=1}^{k} C_j y_{n-j}}, \quad n \in \mathbb{N},$$

$$y_n = \frac{p + \sum_{i=1}^{k} \delta_i x_{n-i} + \sum_{i=1}^{k} \epsilon_i y_{n-i}}{q + \sum_{j=1}^{k} D_j x_{n-j} + \sum_{j=1}^{k} E_j y_{n-j}}, \quad n \in \mathbb{N},$$

with non-negative parameters and non-negative initial conditions. Assume that $A, q > 0$, $I_\gamma \subset I_C$, and there exists a positive integer η, such that for every sequence $\{c_m\}_{m=1}^{\infty}$ with $c_m \in I_\beta$ for $m = 1, 2, \dots$ there exists positive integers, N_1,$N_2 \leq \eta$, such that $\sum_{m=N_1}^{N_2} c_m \in I_B$.
Further assume that there exists a positive integer η_2, such that for every sequence $\{d_m\}_{m=1}^{\infty}$ with $d_m \in I_\epsilon$ for $m = 1, 2, \dots$ there exists positive integers, N_3,$N_4 \leq \eta_2$, such that $\sum_{m=N_3}^{N_4} d_m \in I_E$. Then there exists $M > 0$ and $N \in \mathbb{N}$ so that given any non-negative initial conditions, we have $x_n, y_n \leq M$ for all $n > N$.

Proof. By Theorem 7 we get immediately that there exists $M_1 > 0$ and $N \in \mathbb{N}$ so that given any non-negative initial conditions, we have $x_n \leq M_1$ for all $n > N$. We now use this fact as follows

$$y_n = \frac{p + \sum_{i=1}^{k} \delta_i x_{n-i} + \sum_{i=1}^{k} \epsilon_i y_{n-i}}{q + \sum_{j=1}^{k} D_j x_{n-j} + \sum_{j=1}^{k} E_j y_{n-j}} \leq \frac{p + \sum_{i=1}^{k} \delta_i M_1 + \sum_{i=1}^{k} \epsilon_i y_{n-i}}{q + \sum_{j=1}^{k} E_j y_{n-j}}.$$

Thus we see that the sequence $\{y_n\}_{n=1}^{\infty}$ satisfies the above difference inequality. Moreover using Theorem 1 in [7] we see that there exists $M_2 > 0$ and $N \in \mathbb{N}$ so that given any non-negative initial conditions, we have $y_n \leq M_2$ for all $n > N$.

□

4. Some Boundedness Results Involving a One Sided Comparison

For the results in this section, we either assume that there exists $M_1 > 0$ so that $y_n \leq M_1 x_n$ for all $n \in \mathbb{N}$, or we assume that there exists $M_1 > 0$ and $M_2 \geq 0$ so that $y_n \leq M_1 x_n + M_2$ for all $n \in \mathbb{N}$. We use these one sided comparisons and Theorem 1 of [7] to prove that every solution is bounded for certain systems of rational difference equations.

For the first two results we assume that $y_n \leq M_1 x_n$ for some $M_1 > 0$. No iteration is used in the next result.

Theorem 13. *Suppose that we have a k^{th} order system of two rational difference equations*

$$x_n = \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{n-i} + \sum_{i=1}^{k} \gamma_i y_{n-i}}{A + \sum_{j=1}^{k} B_j x_{n-j} + \sum_{j=1}^{k} C_j y_{n-j}}, \quad n \in \mathbb{N},$$

$$y_n = \frac{p + \sum_{i=1}^{k} \delta_i x_{n-i} + \sum_{i=1}^{k} \epsilon_i y_{n-i}}{q + \sum_{j=1}^{k} D_j x_{n-j} + \sum_{j=1}^{k} E_j y_{n-j}}, \quad n \in \mathbb{N},$$

with non-negative parameters and non-negative initial conditions. Assume that there exists $M_1 > 0$ so that $y_n \leq M_1 x_n$ for all $n \in \mathbb{N}$. Further assume that $A = 0$, $\alpha = 0$, $I_\beta \cup (I_\gamma \setminus I_C) \subset I_B$. Then there exists $M > 0$ and $N \in \mathbb{N}$ so that given any non-negative initial conditions, we have $x_n, y_n \leq M$ for all $n > N$.

Proof. Since we have assumed $y_n \leq M_1 x_n$ it suffices to find a bound for $\{x_n\}_{n=1}^{\infty}$. Notice that

$$x_n = \frac{\sum_{i=1}^{k} \beta_i x_{n-i} + \sum_{i=1}^{k} \gamma_i y_{n-i}}{\sum_{j=1}^{k} B_j x_{n-j} + \sum_{j=1}^{k} C_j y_{n-j}} \leq \sum_{i \in I_C \cap I_\gamma} \frac{\gamma_i}{C_i} + \frac{\sum_{i=1}^{k} \beta_i x_{n-i} + \sum_{i \in I_\gamma \setminus I_C} \gamma_i M_1 x_{n-i}}{\sum_{j=1}^{k} B_j x_{n-j}}$$

$$\leq \sum_{i \in I_C \cap I_\gamma} \frac{\gamma_i}{C_i} + \frac{(1 + M_1) \max_{i \in I_\beta \cup (I_\gamma \setminus I_C)} (\beta_i + \gamma_i)}{\min_{i \in I_\beta \cup (I_\gamma \setminus I_C)} (B_i)}.$$

So $x_n \leq M$ for all $n > N$. Since we have assumed that there exists $M_1 > 0$ so that $y_n \leq M_1 x_n$ for all $n \in \mathbb{N}$, we obtain the full result.

□

In the following result we use Theorem 1 in [7] to iterate, first with respect to y_n and then with respect to x_n.

Theorem 14. *Suppose that we have a k^{th} order system of two rational difference equations*

$$x_n = \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{n-i} + \sum_{i=1}^{k} \gamma_i y_{n-i}}{A + \sum_{j=1}^{k} B_j x_{n-j} + \sum_{j=1}^{k} C_j y_{n-j}}, \quad n \in \mathbb{N},$$

$$y_n = \frac{p + \sum_{i=1}^{k} \delta_i x_{n-i} + \sum_{i=1}^{k} \epsilon_i y_{n-i}}{q + \sum_{j=1}^{k} D_j x_{n-j} + \sum_{j=1}^{k} E_j y_{n-j}}, \quad n \in \mathbb{N},$$

with non-negative parameters and non-negative initial conditions. Assume that there exists $M_1 > 0$ so that $y_n \leq M_1 x_n$ for all $n \in \mathbb{N}$. Further assume that $A = 0$,$q > 0$, one of the following holds

(i) $I_B \subset I_\beta$, $I_C \subset I_\gamma$ and $I_B \neq \emptyset$
(ii) $p > 0$, $I_D \subset I_\delta$, $I_E \subset I_\epsilon$ and $I_C \neq \emptyset$

$I_\delta \subset I_D$, there exists a positive integer η_1, such that for every sequence $\{c_m\}_{m=1}^{\infty}$ with $c_m \in I_\epsilon$ for $m = 1, 2, \ldots$ there exists positive integers, N_1,$N_2 \leq \eta_1$, such that $\sum_{m=N_1}^{N_2} c_m \in I_D \cup I_E$, and there exists a positive integer η_2, such that for every sequence $\{d_m\}_{m=1}^{\infty}$ with $d_m \in I_\beta$ for $m = 1, 2, \ldots$ there exists positive integers, N_3,$N_4 \leq \eta_2$, such that $\sum_{m=N_3}^{N_4} d_m \in I_B$. Then there exists $M > 0$ and $N \in \mathbb{N}$ so that given any non-negative initial conditions, we have $x_n, y_n \leq M$ for all $n > N$.

Proof. First we must prove that $\{y_n\}_{n=1}^{\infty}$ is bounded above. Since $q > 0$ and $I_\delta \subset I_D$ we have,

$$y_n = \frac{p + \sum_{i=1}^{k} \delta_i x_{n-i} + \sum_{i=1}^{k} \epsilon_i y_{n-i}}{q + \sum_{j=1}^{k} D_j x_{n-j} + \sum_{j=1}^{k} E_j y_{n-j}} \leq \frac{p}{q} + \sum_{i \in I_\delta} \frac{\delta_i}{D_i} + \frac{\sum_{i=1}^{k} \epsilon_i y_{n-i}}{q + \sum_{j=1}^{k} D_j x_{n-j} + \sum_{j=1}^{k} E_j y_{n-j}}$$

$$\leq \frac{p}{q} + \sum_{i \in I_\delta} \frac{\delta_i}{D_i} + \frac{\sum_{i=1}^k \epsilon_i y_{n-i}}{q + \sum_{j=1}^k \frac{D_j y_{n-j}}{M_1} + \sum_{j=1}^k E_j y_{n-j}}.$$

Thus the sequence $\{y_n\}_{n=1}^\infty$ satisfies the above difference inequality. Using Theorem 1 in [7] we see that there exists $M_2 > 0$ and $N \in \mathbb{N}$ so that given any non-negative initial conditions, we have $y_n \leq M_2$ for all $n > N$. Let us show that $\{x_n\}$ or $\{y_n\}$ is bounded below by a constant A_2.
Suppose $I_B \subset I_\beta$, $I_C \subset I_\gamma$, and $I_B \neq \emptyset$ then we have

$$x_n = \frac{\alpha + \sum_{i=1}^k \beta_i x_{n-i} + \sum_{i=1}^k \gamma_i y_{n-i}}{\sum_{j=1}^k B_j x_{n-j} + \sum_{j=1}^k C_j y_{n-j}} \geq \frac{\sum_{i \in I_\beta} \beta_i x_{n-i} + \sum_{i \in I_\gamma} \gamma_i y_{n-i}}{\sum_{j \in I_\beta} B_j x_{n-j} + \sum_{j \in I_\gamma} C_j y_{n-j}}$$

$$\geq \frac{\min(\min_{i \in I_\beta}(\beta_i), \min_{i \in I_\gamma}(\gamma_i))}{\max(\max_{j \in I_\beta}(B_j), \max_{j \in I_\gamma}(C_j))}.$$

So in this case $\{x_n\}$ is bounded below by a constant. Now suppose $p > 0$, $I_D \subset I_\delta$, $I_E \subset I_\epsilon$, and $I_C \neq \emptyset$ then we have

$$y_n = \frac{p + \sum_{i=1}^k \delta_i x_{n-i} + \sum_{i=1}^k \epsilon_i y_{n-i}}{q + \sum_{j=1}^k D_j x_{n-j} + \sum_{j=1}^k E_j y_{n-j}} \geq \frac{p + \sum_{i \in I_\delta} \delta_i x_{n-i} + \sum_{i \in I_\epsilon} \epsilon_i y_{n-i}}{q + \sum_{j \in I_\delta} D_j x_{n-j} + \sum_{j \in I_\epsilon} E_j y_{n-j}}$$

$$\geq \frac{\min(p, \min_{i \in I_\delta}(\delta_i), \min_{i \in I_\epsilon}(\epsilon_i))}{\max(q, \max_{j \in I_\delta}(D_j), \max_{j \in I_\epsilon}(E_j))}.$$

So in this case $\{y_n\}$ is bounded below by a constant. Now we have shown that there exists $A_2 > 0$, so that $\{x_n\}$ is bounded below by A_2 and $I_B \neq \emptyset$ or $\{y_n\}$ is bounded below by A_2 and $I_C \neq \emptyset$. We use this fact to show the following.

$$x_n = \frac{\alpha + \sum_{i=1}^k \beta_i x_{n-i} + \sum_{i=1}^k \gamma_i y_{n-i}}{\sum_{j=1}^k B_j x_{n-j} + \sum_{j=1}^k C_j y_{n-j}}$$

$$\leq \frac{\alpha + \sum_{i=1}^k \beta_i x_{n-i} + \sum_{i=1}^k \gamma_i M_2}{\min_+(\sum_{j=1}^k B_j, \sum_{j=1}^k C_j)\frac{A_2}{2} + \frac{1}{2}\sum_{j=1}^k B_j x_{n-j} + \frac{1}{2}\sum_{j=1}^k C_j y_{n-j}}$$

$$\leq \frac{\alpha + \sum_{i=1}^k \beta_i x_{n-i} + \sum_{i=1}^k \gamma_i M_2}{\min_+(\sum_{j=1}^k B_j, \sum_{j=1}^k C_j)\frac{A_2}{2} + \frac{1}{2}\sum_{j=1}^k B_j x_{n-j}}.$$

Thus we see that the sequence $\{x_n\}_{n=1}^\infty$ satisfies the above difference inequality. Using Theorem 1 in [7] we see that there exists $M_3 > 0$ and $N \in \mathbb{N}$ so that given any non-negative initial conditions, we have $x_n \leq M_3$ for all $n > N$.

□

For the next five results we assume that there exists $M_1 > 0$ and $M_2 \geq 0$ so that $y_n \leq M_1 x_n + M_2$ for all $n \in \mathbb{N}$. In the next result we use Theorem 1 of [7] to iterate with respect to y_n and then apply Theorem 11.

Theorem 15. *Suppose that we have a k^{th} order system of two rational difference equations*

$$x_n = \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{n-i} + \sum_{i=1}^{k} \gamma_i y_{n-i}}{A + \sum_{j=1}^{k} B_j x_{n-j} + \sum_{j=1}^{k} C_j y_{n-j}}, \quad n \in \mathbb{N},$$

$$y_n = \frac{p + \sum_{i=1}^{k} \delta_i x_{n-i} + \sum_{i=1}^{k} \epsilon_i y_{n-i}}{q + \sum_{j=1}^{k} D_j x_{n-j} + \sum_{j=1}^{k} E_j y_{n-j}}, \quad n \in \mathbb{N},$$

with non-negative parameters and non-negative initial conditions. Assume that there exists $M_1 > 0$ and $M_2 \geq 0$ so that $y_n \leq M_1 x_n + M_2$ for all $n \in \mathbb{N}$. Further assume that $A, q > 0$ $I_\delta \subset I_D$, there exists a positive integer η_1, such that for every sequence $\{c_m\}_{m=1}^{\infty}$ with $c_m \in I_\epsilon$ for $m = 1, 2, \ldots$ there exists positive integers, N_1,$N_2 \leq \eta_1$, such that $\sum_{m=N_1}^{N_2} c_m \in I_D \cup I_E$, and there exists a positive integer η_2, such that for every sequence $\{d_m\}_{m=1}^{\infty}$ with $d_m \in I_\beta$ for $m = 1, 2, \ldots$ there exists positive integers, N_3,$N_4 \leq \eta_2$, such that $\sum_{m=N_3}^{N_4} d_m \in I_B$. Then there exists $M > 0$ and $N \in \mathbb{N}$ so that given any non-negative initial conditions, we have $x_n, y_n \leq M$ for all $n > N$.

Proof. It suffices to prove that $\{y_n\}_{n=1}^{\infty}$ is bounded above then we simply apply Theorem 11 to obtain the result. Since $q > 0$ and $I_\delta \subset I_D$ we have,

$$y_n = \frac{p + \sum_{i=1}^{k} \delta_i x_{n-i} + \sum_{i=1}^{k} \epsilon_i y_{n-i}}{q + \sum_{j=1}^{k} D_j x_{n-j} + \sum_{j=1}^{k} E_j y_{n-j}} \leq \frac{p}{q} + \sum_{i \in I_\delta} \frac{\delta_i}{D_i} + \frac{\sum_{i=1}^{k} \epsilon_i y_{n-i}}{q + \sum_{j=1}^{k} D_j x_{n-j} + \sum_{j=1}^{k} E_j y_{n-j}}$$

$$\leq \frac{p}{q} + \sum_{i \in I_\delta} \frac{\delta_i}{D_i} + \frac{\sum_{i=1}^{k} \epsilon_i y_{n-i}}{\frac{q}{2} + \sum_{j=1}^{k} F_j y_{n-j} + \sum_{j=1}^{k} E_j y_{n-j}}.$$

Where $F_j = \min(\frac{D_j}{M_1}, \frac{q}{2k(M_2+1)})$. Thus the sequence $\{y_n\}_{n=1}^{\infty}$ satisfies the above difference inequality. Using Theorem 1 in [7] we see that there exists $M > 0$ and $N \in \mathbb{N}$ so that given any non-negative initial conditions, we have $y_n \leq M$ for all $n > N$. Thus applying Theorem 11 we obtain the result.

□

For the following result we use Theorem 1 of [7] to iterate with respect to x_n and then we use the inequality $y_n \leq M_1 x_n + M_2$ for $M_1 > 0$ and $M_2 \geq 0$.

Theorem 16. *Suppose that we have a k^{th} order system of two rational difference equations*

$$x_n = \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{n-i} + \sum_{i=1}^{k} \gamma_i y_{n-i}}{A + \sum_{j=1}^{k} B_j x_{n-j} + \sum_{j=1}^{k} C_j y_{n-j}}, \quad n \in \mathbb{N},$$

$$y_n = \frac{p + \sum_{i=1}^{k} \delta_i x_{n-i} + \sum_{i=1}^{k} \epsilon_i y_{n-i}}{q + \sum_{j=1}^{k} D_j x_{n-j} + \sum_{j=1}^{k} E_j y_{n-j}}, \quad n \in \mathbb{N},$$

with non-negative parameters and non-negative initial conditions. Assume that there exists $M_1 > 0$ and $M_2 \geq 0$ so that $y_n \leq M_1 x_n + M_2$ for all $n \in \mathbb{N}$. Further assume that $A > 0$ there exists a positive integer η, such that for every sequence $\{c_m\}_{m=1}^{\infty}$ with $c_m \in I_\beta \cup (I_\gamma \setminus I_C)$ for $m = 1, 2, \ldots$ there exists positive integers, N_1,$N_2 \leq \eta$, such that

$\sum_{m=N_1}^{N_2} c_m \in I_B$. *Then there exists $M>0$ and $N\in\mathbb{N}$ so that given any non-negative initial conditions, we have $x_n, y_n \leq M$ for all $n>N$.*

Proof. Since we have assumed $y_n \leq M_1x_n + M_2$ it suffices to find a bound for $\{x_n\}_{n=1}^{\infty}$. Notice that

$$x_n = \frac{\alpha+\sum_{i=1}^{k}\beta_i x_{n-i}+\sum_{i=1}^{k}\gamma_i y_{n-i}}{A+\sum_{j=1}^{k}B_j x_{n-j}+\sum_{j=1}^{k}C_j y_{n-j}} \leq \frac{\alpha}{A}+\sum_{i\in I_C\cap I_\gamma}\frac{\gamma_i}{C_i}+\frac{\sum_{i=1}^{k}\beta_i x_{n-i}+\sum_{i\in I_\gamma\setminus I_C}\gamma_i y_{n-i}}{A+\sum_{j=1}^{k}B_j x_{n-j}}$$

$$\leq \frac{p}{q}+\sum_{i\in I_C\cap I_\gamma}\frac{\gamma_i}{C_i}+\frac{\sum_{i\in I_\gamma\setminus I_C}\gamma_i M_2+\sum_{i=1}^{k}\beta_i x_{n-i}+\sum_{i\in I_\gamma\setminus I_C}\gamma_i M_1 x_{n-i}}{A+\sum_{j=1}^{k}B_j x_{n-j}}.$$

Thus the sequence $\{x_n\}_{n=1}^{\infty}$ satisfies the above difference inequality. Using Theorem 1 in [7] we see that there exists $M>0$ and $N\in\mathbb{N}$ so that given any non-negative initial conditions, we have $x_n\leq M$ for all $n>N$. Since there exists $M_1>0$ and $M_2\geq 0$ so that $y_n\leq M_1x_n+M_2$ for all $n\in\mathbb{N}$, we obtain the full result.

□

For the next result we show that x_n or y_n is bounded below and then we use Theorem 1 of [7] to iterate with respect to x_n. Once we have shown x_n is bounded above we use the fact that $y_n\leq M_1x_n+M_2$ for $M_1>0$ and $M_2\geq 0$ to obtain the result.

Theorem 17. *Suppose that we have a k^{th} order system of two rational difference equations*

$$x_n = \frac{\alpha+\sum_{i=1}^{k}\beta_i x_{n-i}+\sum_{i=1}^{k}\gamma_i y_{n-i}}{A+\sum_{j=1}^{k}B_j x_{n-j}+\sum_{j=1}^{k}C_j y_{n-j}}, \quad n\in\mathbb{N},$$
$$y_n = \frac{p+\sum_{i=1}^{k}\delta_i x_{n-i}+\sum_{i=1}^{k}\epsilon_i y_{n-i}}{q+\sum_{j=1}^{k}D_j x_{n-j}+\sum_{j=1}^{k}E_j y_{n-j}}, \quad n\in\mathbb{N},$$

with non-negative parameters and non-negative initial conditions. Assume that there exists $M_1>0$ and $M_2\geq 0$ so that $y_n\leq M_1x_n+M_2$ for all $n\in\mathbb{N}$. Further assume that $A=0$, one of the following holds

(i) $I_B\subset I_\beta$, $I_C\subset I_\gamma$ and $I_B\neq\emptyset$
(ii) $q=0$, $I_D\subset I_\delta$, $I_E\subset I_\epsilon$ and $I_C\neq\emptyset$
(iii) $p,q>0$, $I_D\subset I_\delta$, $I_E\subset I_\epsilon$ and $I_C\neq\emptyset$

and there exists a positive integer η, such that for every sequence $\{c_m\}_{m=1}^{\infty}$ with $c_m\in I_\beta\cup(I_\gamma\setminus I_C)$ for $m=1,2,\ldots$ there exists positive integers, N_1,$N_2\leq\eta$, such that $\sum_{m=N_1}^{N_2}c_m\in I_B$. Then there exists $M>0$ and $N\in\mathbb{N}$ so that given any non-negative initial conditions, we have $x_n,y_n\leq M$ for all $n>N$.

Proof. Suppose $I_B\subset I_\beta$, $I_C\subset I_\gamma$, and $I_B\neq\emptyset$ then we have

$$x_n = \frac{\alpha+\sum_{i=1}^{k}\beta_i x_{n-i}+\sum_{i=1}^{k}\gamma_i y_{n-i}}{\sum_{j=1}^{k}B_j x_{n-j}+\sum_{j=1}^{k}C_j y_{n-j}} \geq \frac{\sum_{i\in I_\beta}\beta_i x_{n-i}+\sum_{i\in I_\gamma}\gamma_i y_{n-i}}{\sum_{j\in I_\beta}B_j x_{n-j}+\sum_{j\in I_\gamma}C_j y_{n-j}}$$
$$\geq \frac{\min(\min_{i\in I_\beta}(\beta_i),\min_{i\in I_\gamma}(\gamma_i))}{\max(\max_{j\in I_\beta}(B_j),\max_{j\in I_\gamma}(C_j))}.$$

So in this case $\{x_n\}$ is bounded below by a constant. Now suppose $q = 0$, $I_D \subset I_\delta$, $I_E \subset I_\epsilon$, and $I_C \neq \emptyset$ then we have

$$y_n = \frac{p + \sum_{i=1}^{k} \delta_i x_{n-i} + \sum_{i=1}^{k} \epsilon_i y_{n-i}}{\sum_{j=1}^{k} D_j x_{n-j} + \sum_{j=1}^{k} E_j y_{n-j}} \geq \frac{\sum_{i\in I_\delta} \delta_i x_{n-i} + \sum_{i\in I_\epsilon} \epsilon_i y_{n-i}}{\sum_{j\in I_\delta} D_j x_{n-j} + \sum_{j\in I_\epsilon} E_j y_{n-j}}$$
$$\geq \frac{\min(\min_{i\in I_\delta}(\delta_i), \min_{i\in I_\epsilon}(\epsilon_i))}{\max(\max_{j\in I_\delta}(D_j), \max_{j\in I_\epsilon}(E_j))}.$$

So in this case $\{y_n\}$ is bounded below by a constant. Now suppose $p, q > 0$, $I_D \subset I_\delta$, $I_E \subset I_\epsilon$, and $I_C \neq \emptyset$ then we have

$$y_n = \frac{p + \sum_{i=1}^{k} \delta_i x_{n-i} + \sum_{i=1}^{k} \epsilon_i y_{n-i}}{q + \sum_{j=1}^{k} D_j x_{n-j} + \sum_{j=1}^{k} E_j y_{n-j}} \geq \frac{p + \sum_{i\in I_\delta} \delta_i x_{n-i} + \sum_{i\in I_\epsilon} \epsilon_i y_{n-i}}{q + \sum_{j\in I_\delta} D_j x_{n-j} + \sum_{j\in I_\epsilon} E_j y_{n-j}}$$
$$\geq \frac{\min(p, \min_{i\in I_\delta}(\delta_i), \min_{i\in I_\epsilon}(\epsilon_i))}{\max(q, \max_{j\in I_\delta}(D_j), \max_{j\in I_\epsilon}(E_j))}.$$

So in this case $\{y_n\}$ is bounded below by a constant. Now we have shown that there exists $A_2 > 0$, so that $\{x_n\}$ is bounded below by A_2 and $I_B \neq \emptyset$ or $\{y_n\}$ is bounded below by A_2 and $I_C \neq \emptyset$. We use this fact to show the following.

$$x_n = \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{n-i} + \sum_{i=1}^{k} \gamma_i y_{n-i}}{\sum_{j=1}^{k} B_j x_{n-j} + \sum_{j=1}^{k} C_j y_{n-j}} \leq \frac{\sum_{i\in I_\gamma \cap I_C} \gamma_i y_{n-i}}{\sum_{j=1}^{k} B_j x_{n-j} + \sum_{j=1}^{k} C_j y_{n-j}}$$
$$+ \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{n-i} + \sum_{i\in I_\gamma \setminus I_C} \gamma_i y_{n-i}}{\min_+(\sum_{j=1}^{k} B_j, \sum_{j=1}^{k} C_j)\frac{A_2}{2} + \frac{1}{2}\sum_{j=1}^{k} B_j x_{n-j} + \frac{1}{2}\sum_{j=1}^{k} C_j y_{n-j}}$$
$$\leq \sum_{i\in I_\gamma \cap I_C} \frac{\gamma_i}{C_i} + \frac{\alpha + \sum_{i\in I_\gamma\setminus I_C} \gamma_i M_2 + \sum_{i=1}^{k} \beta_i x_{n-i} + \sum_{i\in I_\gamma \setminus I_C} \gamma_i M_1 x_{n-i}}{\min_+(\sum_{j=1}^{k} B_j, \sum_{j=1}^{k} C_j)\frac{A_2}{2} + \frac{1}{2}\sum_{j=1}^{k} B_j x_{n-j}}.$$

Thus we see that the sequence $\{x_n\}_{n=1}^{\infty}$ satisfies the above difference inequality. Using Theorem 1 in [7] we see that there exists $M > 0$ and $N \in \mathbb{N}$ so that given any non-negative initial conditions, we have $x_n \leq M$ for all $n > N$. Since there exists $M_1 > 0$ and $M_2 \geq 0$ so that $y_n \leq M_1 x_n + M_2$ for all $n \in \mathbb{N}$, we have the result.

□

In the following theorem, we first use Theorem 1 from [7] to iterate with respect to y_n. We then use the fact that y_n is bounded above along with our assumptions to show that x_n or y_n is bounded below. We then use these facts along with Theorem 1 from [7] to iterate with respect to x_n.

Theorem 18. *Suppose that we have a k^{th} order system of two rational difference equations*

$$x_n = \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{n-i} + \sum_{i=1}^{k} \gamma_i y_{n-i}}{A + \sum_{j=1}^{k} B_j x_{n-j} + \sum_{j=1}^{k} C_j y_{n-j}}, \quad n \in \mathbb{N},$$
$$y_n = \frac{p + \sum_{i=1}^{k} \delta_i x_{n-i} + \sum_{i=1}^{k} \epsilon_i y_{n-i}}{q + \sum_{j=1}^{k} D_j x_{n-j} + \sum_{j=1}^{k} E_j y_{n-j}}, \quad n \in \mathbb{N},$$

with non-negative parameters and non-negative initial conditions. Assume that there exists $M_1 > 0$ and $M_2 \geq 0$ so that $y_n \leq M_1 x_n + M_2$ for all $n \in \mathbb{N}$. Further assume that $A = 0$,$q > 0$, one of the following holds

(i) $I_B \subset I_\beta$, $I_C \subset I_\gamma$ *and* $I_B \neq \emptyset$

(ii) $p > 0$, $I_D \subset I_\delta$, $I_E \subset I_\epsilon$ *and* $I_C \neq \emptyset$

$I_\delta \subset I_D$*, there exists a positive integer* η_1*, such that for every sequence* $\{c_m\}_{m=1}^\infty$ *with* $c_m \in I_\epsilon$ *for* $m = 1, 2, \ldots$ *there exists positive integers,* N_1,$N_2 \leq \eta_1$ *, such that* $\sum_{m=N_1}^{N_2} c_m \in I_D \cup I_E$*, and there exists a positive integer* η_2*, such that for every sequence* $\{d_m\}_{m=1}^\infty$ *with* $d_m \in I_\beta$ *for* $m = 1, 2, \ldots$ *there exists positive integers,* N_3,$N_4 \leq \eta_2$ *, such that* $\sum_{m=N_3}^{N_4} d_m \in I_B$*. Then there exists* $M > 0$ *and* $N \in \mathbb{N}$ *so that given any non-negative initial conditions, we have* $x_n, y_n \leq M$ *for all* $n > N$*.*

Proof. First we must prove that $\{y_n\}_{n=1}^\infty$ is bounded above. Since $q > 0$ and $I_\delta \subset I_D$ we have,

$$y_n = \frac{p + \sum_{i=1}^k \delta_i x_{n-i} + \sum_{i=1}^k \epsilon_i y_{n-i}}{q + \sum_{j=1}^k D_j x_{n-j} + \sum_{j=1}^k E_j y_{n-j}} \leq \frac{p}{q} + \sum_{i \in I_\delta} \frac{\delta_i}{D_i} + \frac{\sum_{i=1}^k \epsilon_i y_{n-i}}{q + \sum_{j=1}^k D_j x_{n-j} + \sum_{j=1}^k E_j y_{n-j}}$$

$$\leq \frac{p}{q} + \sum_{i \in I_\delta} \frac{\delta_i}{D_i} + \frac{\sum_{i=1}^k \epsilon_i y_{n-i}}{\frac{q}{2} + \sum_{j=1}^k F_j y_{n-j} + \sum_{j=1}^k E_j y_{n-j}},$$

where $F_j = \min(\frac{D_j}{M_1}, \frac{q}{2k(M_2+1)})$. Thus the sequence $\{y_n\}_{n=1}^\infty$ satisfies the above difference inequality. Using Theorem 1 in [7] we see that there exists $M_3 > 0$ and $N \in \mathbb{N}$ so that given any non-negative initial conditions, we have $y_n \leq M_3$ for all $n > N$. Let us show that $\{x_n\}$ or $\{y_n\}$ is bounded below by a constant A_2.

Suppose $I_B \subset I_\beta$, $I_C \subset I_\gamma$, and $I_B \neq \emptyset$ then we have

$$x_n = \frac{\alpha + \sum_{i=1}^k \beta_i x_{n-i} + \sum_{i=1}^k \gamma_i y_{n-i}}{\sum_{j=1}^k B_j x_{n-j} + \sum_{j=1}^k C_j y_{n-j}} \geq \frac{\sum_{i \in I_\beta} \beta_i x_{n-i} + \sum_{i \in I_\gamma} \gamma_i y_{n-i}}{\sum_{j \in I_\beta} B_j x_{n-j} + \sum_{j \in I_\gamma} C_j y_{n-j}}$$

$$\geq \frac{\min(\min_{i \in I_\beta}(\beta_i), \min_{i \in I_\gamma}(\gamma_i))}{\max(\max_{j \in I_\beta}(B_j), \max_{j \in I_\gamma}(C_j))}.$$

So in this case $\{x_n\}$ is bounded below by a constant. Now suppose $p > 0$, $I_D \subset I_\delta$, $I_E \subset I_\epsilon$, and $I_C \neq \emptyset$ then we have

$$y_n = \frac{p + \sum_{i=1}^k \delta_i x_{n-i} + \sum_{i=1}^k \epsilon_i y_{n-i}}{q + \sum_{j=1}^k D_j x_{n-j} + \sum_{j=1}^k E_j y_{n-j}} \geq \frac{p + \sum_{i \in I_\delta} \delta_i x_{n-i} + \sum_{i \in I_\epsilon} \epsilon_i y_{n-i}}{q + \sum_{j \in I_\delta} D_j x_{n-j} + \sum_{j \in I_\epsilon} E_j y_{n-j}}$$

$$\geq \frac{\min(p, \min_{i \in I_\delta}(\delta_i), \min_{i \in I_\epsilon}(\epsilon_i))}{\max(q, \max_{j \in I_\delta}(D_j), \max_{j \in I_\epsilon}(E_j))}.$$

So in this case $\{y_n\}$ is bounded below by a constant. Now we have shown that there exists $A_2 > 0$, so that $\{x_n\}$ is bounded below by A_2 and $I_B \neq \emptyset$ or $\{y_n\}$ is bounded

below by A_2 and $I_C \neq \emptyset$. We use this fact to show the following.

$$x_n = \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{n-i} + \sum_{i=1}^{k} \gamma_i y_{n-i}}{\sum_{j=1}^{k} B_j x_{n-j} + \sum_{j=1}^{k} C_j y_{n-j}}$$

$$\leq \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{n-i} + \sum_{i=1}^{k} \gamma_i M_3}{\min_+(\sum_{j=1}^{k} B_j, \sum_{j=1}^{k} C_j)\frac{A_2}{2} + \frac{1}{2}\sum_{j=1}^{k} B_j x_{n-j} + \frac{1}{2}\sum_{j=1}^{k} C_j y_{n-j}}$$

$$\leq \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{n-i} + \sum_{i=1}^{k} \gamma_i M_3}{\min_+(\sum_{j=1}^{k} B_j, \sum_{j=1}^{k} C_j)\frac{A_2}{2} + \frac{1}{2}\sum_{j=1}^{k} B_j x_{n-j}}.$$

Thus we see that the sequence $\{x_n\}_{n=1}^{\infty}$ satisfies the above difference inequality. Using Theorem 1 in [7] we see that there exists $M_4 > 0$ and $N \in \mathbb{N}$ so that given any non-negative initial conditions, we have $x_n \leq M_4$ for all $n > N$.

□

In the following theorem, we first show that x_n or y_n is bounded below. We then use Theorem 1 from [7] to iterate with respect to y_n. We then use the fact that y_n is bounded above and we use the fact that x_n or y_n is bounded below along with Theorem 1 from [7] to iterate with respect to x_n.

Theorem 19. *Suppose that we have a k^{th} order system of two rational difference equations*

$$x_n = \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{n-i} + \sum_{i=1}^{k} \gamma_i y_{n-i}}{A + \sum_{j=1}^{k} B_j x_{n-j} + \sum_{j=1}^{k} C_j y_{n-j}}, \quad n \in \mathbb{N},$$

$$y_n = \frac{p + \sum_{i=1}^{k} \delta_i x_{n-i} + \sum_{i=1}^{k} \epsilon_i y_{n-i}}{q + \sum_{j=1}^{k} D_j x_{n-j} + \sum_{j=1}^{k} E_j y_{n-j}}, \quad n \in \mathbb{N},$$

with non-negative parameters and non-negative initial conditions. Assume that there exists $M_1 > 0$ and $M_2 \geq 0$ so that $y_n \leq M_1 x_n + M_2$ for all $n \in \mathbb{N}$. Further assume that $A = 0$,$q = 0$, one of the following holds

(i) $I_B \subset I_\beta$, $I_C \subset I_\gamma$ and $I_B, I_D \neq \emptyset$
(ii) $I_D \subset I_\delta$, $I_E \subset I_\epsilon$ and $I_C, I_E \neq \emptyset$

$I_\delta \subset I_D$, there exists a positive integer η_1, such that for every sequence $\{c_m\}_{m=1}^{\infty}$ with $c_m \in I_\epsilon$ for $m = 1, 2, \ldots$ there exists positive integers, N_1,$N_2 \leq \eta_1$, such that $\sum_{m=N_1}^{N_2} c_m \in I_D \cup I_E$, and there exists a positive integer η_2, such that for every sequence $\{d_m\}_{m=1}^{\infty}$ with $d_m \in I_\beta$ for $m = 1, 2, \ldots$ there exists positive integers, N_3,$N_4 \leq \eta_2$, such that $\sum_{m=N_3}^{N_4} d_m \in I_B$. Then there exists $M > 0$ and $N \in \mathbb{N}$ so that given any non-negative initial conditions, we have $x_n, y_n \leq M$ for all $n > N$.

Proof. Suppose $I_B \subset I_\beta$, $I_C \subset I_\gamma$ and $I_B, I_D \neq \emptyset$ then we have

$$x_n = \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{n-i} + \sum_{i=1}^{k} \gamma_i y_{n-i}}{\sum_{j=1}^{k} B_j x_{n-j} + \sum_{j=1}^{k} C_j y_{n-j}} \geq \frac{\sum_{i\in I_\beta} \beta_i x_{n-i} + \sum_{i\in I_\gamma} \gamma_i y_{n-i}}{\sum_{j\in I_\beta} B_j x_{n-j} + \sum_{j\in I_\gamma} C_j y_{n-j}}$$

$$\geq \frac{\min(\min_{i\in I_\beta}(\beta_i), \min_{i\in I_\gamma}(\gamma_i))}{\max(\max_{j\in I_\beta}(B_j), \max_{j\in I_\gamma}(C_j))}.$$

So in this case $\{x_n\}$ is bounded below by a constant. Now suppose $I_D \subset I_\delta$, $I_E \subset I_\epsilon$ and $I_C, I_E \neq \emptyset$ then we have

$$y_n = \frac{p + \sum_{i=1}^k \delta_i x_{n-i} + \sum_{i=1}^k \epsilon_i y_{n-i}}{\sum_{j=1}^k D_j x_{n-j} + \sum_{j=1}^k E_j y_{n-j}} \geq \frac{\sum_{i\in I_\delta} \delta_i x_{n-i} + \sum_{i\in I_\epsilon} \epsilon_i y_{n-i}}{\sum_{j\in I_\delta} D_j x_{n-j} + \sum_{j\in I_\epsilon} E_j y_{n-j}}$$

$$\geq \frac{\min(\min_{i\in I_\delta}(\delta_i), \min_{i\in I_\epsilon}(\epsilon_i))}{\max(\max_{j\in I_\delta}(D_j), \max_{j\in I_\epsilon}(E_j))}.$$

So in this case $\{y_n\}$ is bounded below by a constant. Now we have shown that there exists $A_2 > 0$, so that $\{x_n\}$ is bounded below by A_2 and $I_D \neq \emptyset$ or $\{y_n\}$ is bounded below by A_2 and $I_E \neq \emptyset$. We use this fact to show that $\{y_n\}$ is bounded above by a constant M_3.

$$y_n = \frac{p + \sum_{i=1}^k \delta_i x_{n-i} + \sum_{i=1}^k \epsilon_i y_{n-i}}{\sum_{j=1}^k D_j x_{n-j} + \sum_{j=1}^k E_j y_{n-j}}$$

$$\leq \frac{p}{\min_+(\sum_{j=1}^k D_j, \sum_{j=1}^k E_j)A_2} + \sum_{i\in I_\delta} \frac{\delta_i}{D_i}$$

$$+2\frac{\sum_{i=1}^k \epsilon_i y_{n-i}}{\min_+(\sum_{j=1}^k D_j, \sum_{j=1}^k E_j)A_2 + \sum_{j=1}^k D_j x_{n-j} + \sum_{j=1}^k E_j y_{n-j}}$$

$$\leq \frac{p}{\min_+(\sum_{j=1}^k D_j, \sum_{j=1}^k E_j)A_2} + \sum_{i\in I_\delta} \frac{\delta_i}{D_i}$$

$$+2\frac{\sum_{i=1}^k \epsilon_i y_{n-i}}{\min_+(\sum_{j=1}^k D_j, \sum_{j=1}^k E_j)\frac{A_2}{2} + \sum_{j=1}^k F_j y_{n-j} + \sum_{j=1}^k E_j y_{n-j}},$$

where $F_j = \min(\frac{D_j}{M_1}, \frac{\min_+(\sum_{j=1}^k D_j, \sum_{j=1}^k E_j)A_2}{2k(M_2+1)})$. Thus the sequence $\{y_n\}_{n=1}^\infty$ satisfies the above difference inequality. Using Theorem 1 in [7] we see that there exists $M_3 > 0$ and $N \in \mathbb{N}$ so that given any non-negative initial conditions, we have $y_n \leq M_3$ for all $n > N$. Now we have shown that there exists $A_2 > 0$, so that $\{x_n\}$ is bounded below by A_2 and $I_B \neq \emptyset$ or $\{y_n\}$ is bounded below by A_2 and $I_C \neq \emptyset$. Finally we use this fact to show that $\{x_n\}$ is bounded above by a constant M_4.

$$x_n = \frac{\alpha + \sum_{i=1}^k \beta_i x_{n-i} + \sum_{i=1}^k \gamma_i y_{n-i}}{\sum_{j=1}^k B_j x_{n-j} + \sum_{j=1}^k C_j y_{n-j}}$$

$$\leq \frac{\alpha + \sum_{i=1}^k \beta_i x_{n-i} + \sum_{i=1}^k \gamma_i M_3}{\min_+(\sum_{j=1}^k B_j, \sum_{j=1}^k C_j)\frac{A_2}{2} + \frac{1}{2}\sum_{j=1}^k B_j x_{n-j} + \frac{1}{2}\sum_{j=1}^k C_j y_{n-j}}$$

$$\leq \frac{\alpha + \sum_{i=1}^k \beta_i x_{n-i} + \sum_{i=1}^k \gamma_i M_3}{\min_+(\sum_{j=1}^k B_j, \sum_{j=1}^k C_j)\frac{A_2}{2} + \frac{1}{2}\sum_{j=1}^k B_j x_{n-j}}.$$

Thus we see that the sequence $\{x_n\}_{n=1}^\infty$ satisfies the above difference inequality. Using Theorem 1 in [7] we see that there exists $M_4 > 0$ and $N \in \mathbb{N}$ so that given any non-negative initial conditions, we have $x_n \leq M_4$ for all $n > N$.

$\square$

5. A Comparison on Both Sides Involving Constants

For the results in this section, we assume that there exists $M_1, M_3 > 0$ and $M_4 \geq M_2 \geq 0$ so that $x_n \leq M_1 y_n + M_2 \leq M_3 x_n + M_4$ for all $n \in \mathbb{N}$. We use this comparison and Theorem 1 of [7] to prove that every solution is bounded for certain systems of rational difference equations. For the following result we use Theorem 1 of [7] to iterate with respect to x_n and then we use the fact that there exists $M_1, M_3 > 0$ and $M_4 \geq M_2 \geq 0$ so that $x_n \leq M_1 y_n + M_2 \leq M_3 x_n + M_4$ for all $n \in \mathbb{N}$.

Theorem 20. *Suppose that we have a k^{th} order system of two rational difference equations*

$$x_n = \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{n-i} + \sum_{i=1}^{k} \gamma_i y_{n-i}}{A + \sum_{j=1}^{k} B_j x_{n-j} + \sum_{j=1}^{k} C_j y_{n-j}}, \quad n \in \mathbb{N},$$

$$y_n = \frac{p + \sum_{i=1}^{k} \delta_i x_{n-i} + \sum_{i=1}^{k} \epsilon_i y_{n-i}}{q + \sum_{j=1}^{k} D_j x_{n-j} + \sum_{j=1}^{k} E_j y_{n-j}}, \quad n \in \mathbb{N},$$

with non-negative parameters and non-negative initial conditions. Further assume that there exists $M_1, M_3 > 0$ and $M_4 \geq M_2 \geq 0$ so that $x_n \leq M_1 y_n + M_2 \leq M_3 x_n + M_4$ for all $n \in \mathbb{N}$, and suppose that $A > 0$ and there exists a positive integer η, such that for every sequence $\{c_m\}_{m=1}^{\infty}$ with $c_m \in I_\beta \cup I_\gamma$ for $m = 1, 2, \dots$ there exists positive integers, $N_1, N_2 \leq \eta$, such that $\sum_{m=N_1}^{N_2} c_m \in I_B \cup I_C$. Then there exists $M > 0$ and $N \in \mathbb{N}$ so that given any non-negative initial conditions, we have $x_n, y_n \leq M$ for all $n > N$.

Proof. We have

$$\begin{aligned} x_n &= \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{n-i} + \sum_{i=1}^{k} \gamma_i y_{n-i}}{A + \sum_{j=1}^{k} B_j x_{n-j} + \sum_{j=1}^{k} C_j y_{n-j}} \\ &\leq \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{n-i} + (\frac{M_3}{M_1}) \sum_{i=1}^{k} \gamma_i x_{n-i} + (\frac{M_4 - M_2}{M_1}) \sum_{i=1}^{k} \gamma_i}{\frac{A}{2} + \sum_{j=1}^{k} B_j x_{n-j} + \sum_{j=1}^{k} F_j x_{n-j}}, \end{aligned}$$

where $F_j = \min(\frac{C_j}{M_1}, \frac{A}{2k(M_2+1)})$. Thus we see that the sequence $\{x_n\}_{n=1}^{\infty}$ satisfies the above difference inequality. Moreover using Theorem 1 in [7] we see that there exists $M > 0$ and $N \in \mathbb{N}$ so that given any non-negative initial conditions, we have $x_n \leq M$ for all $n > N$. Since there exists $M_1, M_3 > 0$ and $M_4 \geq M_2 \geq 0$ so that $x_n \leq M_1 y_n + M_2 \leq M_3 x_n + M_4$ for all $n \in \mathbb{N}$, we have the full result.

□

For the following theorem we use Theorem 1 of [7] to iterate with respect to y_n and then we use the fact that there exists $M_1, M_3 > 0$ and $M_4 \geq M_2 \geq 0$ so that $x_n \leq M_1 y_n + M_2 \leq M_3 x_n + M_4$ for all $n \in \mathbb{N}$.

Theorem 21. *Suppose that we have a k^{th} order system of two rational difference equations*

$$x_n = \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{n-i} + \sum_{i=1}^{k} \gamma_i y_{n-i}}{A + \sum_{j=1}^{k} B_j x_{n-j} + \sum_{j=1}^{k} C_j y_{n-j}}, \quad n \in \mathbb{N},$$

$$y_n = \frac{p+\sum_{i=1}^{k}\delta_i x_{n-i}+\sum_{i=1}^{k}\epsilon_i y_{n-i}}{q+\sum_{j=1}^{k}D_j x_{n-j}+\sum_{j=1}^{k}E_j y_{n-j}}, \quad n\in\mathbb{N},$$

with non-negative parameters and non-negative initial conditions. Further assume that there exists $M_1, M_3 > 0$ *and* $M_4 \geq M_2 \geq 0$ *so that* $x_n \leq M_1 y_n + M_2 \leq M_3 x_n + M_4$ *for all* $n\in\mathbb{N}$, *and suppose that* $q > 0$ *and there exists a positive integer* η, *such that for every sequence* $\{c_m\}_{m=1}^{\infty}$ *with* $c_m \in I_\delta \cup I_\epsilon$ *for* $m = 1, 2, \dots$ *there exists positive integers,* N_1,$N_2 \leq \eta$, *such that* $\sum_{m=N_1}^{N_2} c_m \in I_D \cup I_E$. *Then there exists* $M > 0$ *and* $N\in\mathbb{N}$ *so that given any non-negative initial conditions, we have* $x_n, y_n \leq M$ *for all* $n > N$.

Proof. We have

$$y_n = \frac{p+\sum_{i=1}^{k}\delta_i x_{n-i}+\sum_{i=1}^{k}\epsilon_i y_{n-i}}{q+\sum_{j=1}^{k}D_j x_{n-j}+\sum_{j=1}^{k}E_j y_{n-j}} \leq \frac{p+\sum_{i=1}^{k}\delta_i M_2+\sum_{i=1}^{k}\delta_i M_1 y_{n-i}+\sum_{i=1}^{k}\epsilon_i y_{n-i}}{\frac{q}{2}+\sum_{j=1}^{k}G_j y_{n-j}+\sum_{j=1}^{k}E_j y_{n-j}},$$

where $G_j = \min(\frac{D_j M_1}{M_3}, \frac{qM_1}{2k(M_4-M_2+1)})$. Thus we see that the sequence $\{y_n\}_{n=1}^{\infty}$ satisfies the above difference inequality. Again using Theorem 1 in [7] we see that there exists $M > 0$ and $N\in\mathbb{N}$ so that given any non-negative initial conditions, we have $y_n \leq M$ for all $n > N$. Since there exists $M_1, M_3 > 0$ and $M_4 \geq M_2 \geq 0$ so that $x_n \leq M_1 y_n + M_2 \leq M_3 x_n + M_4$ for all $n\in\mathbb{N}$, we have the full result.

□

For the following result we first show that x_n or y_n is bounded below by a constant. Then we use Theorem 1 of [7] to iterate with respect to x_n. The fact that there exists $M_1, M_3 > 0$ and $M_4 \geq M_2 \geq 0$ so that $x_n \leq M_1 y_n + M_2 \leq M_3 x_n + M_4$ for all $n\in\mathbb{N}$ gives us the full result.

Theorem 22. *Suppose that we have a* k^{th} *order system of two rational difference equations*

$$x_n = \frac{\alpha+\sum_{i=1}^{k}\beta_i x_{n-i}+\sum_{i=1}^{k}\gamma_i y_{n-i}}{A+\sum_{j=1}^{k}B_j x_{n-j}+\sum_{j=1}^{k}C_j y_{n-j}}, \quad n\in\mathbb{N},$$

$$y_n = \frac{p+\sum_{i=1}^{k}\delta_i x_{n-i}+\sum_{i=1}^{k}\epsilon_i y_{n-i}}{q+\sum_{j=1}^{k}D_j x_{n-j}+\sum_{j=1}^{k}E_j y_{n-j}}, \quad n\in\mathbb{N},$$

with non-negative parameters and non-negative initial conditions. Further assume that there exists $M_1, M_3 > 0$ *and* $M_4 > M_2 > 0$ *so that* $x_n \leq M_1 y_n + M_2 \leq M_3 x_n + M_4$ *for all* $n\in\mathbb{N}$, *and suppose that* $A = 0$ *and one of the following holds*

(i) $I_B \cup I_C \subset I_\beta \cup I_\gamma$, $\alpha > 0$, *and* $I_B \neq \emptyset$
(ii) $I_D \cup I_E \subset I_\delta \cup I_\epsilon$, $p > 0$, *and* $I_C \neq \emptyset$

and there exists a positive integer η, *such that for every sequence* $\{c_m\}_{m=1}^{\infty}$ *with* $c_m \in I_\beta \cup I_\gamma$ *for* $m = 1, 2, \dots$ *there exists positive integers,* N_1,$N_2 \leq \eta$, *such that* $\sum_{m=N_1}^{N_2} c_m \in I_B \cup I_C$. *Then there exists* $M > 0$ *and* $N\in\mathbb{N}$ *so that given any non-negative initial conditions, we have* $x_n, y_n \leq M$ *for all* $n > N$.

Proof. Suppose that $I_B \cup I_C \subset I_\beta \cup I_\gamma$, $\alpha > 0$, and $I_B \neq \emptyset$ then we have

$$x_n = \frac{\alpha + \sum_{i=1}^k \beta_i x_{n-i} + \sum_{i=1}^k \gamma_i y_{n-i}}{\sum_{j=1}^k B_j x_{n-j} + \sum_{j=1}^k C_j y_{n-j}} \geq \frac{\frac{\alpha}{2} + \sum_{i \in I_\beta} H_i y_{n-i} + \sum_{i \in I_\gamma} \gamma_i y_{n-i}}{\sum_{j \in I_\beta \cup I_\gamma} (B_j M_2 + B_j M_1 y_{n-j} + C_j y_{n-j})}$$
$$\geq \frac{\min(\frac{\alpha}{2}, \min_{i \in I_\beta}(H_i), \min_{i \in I_\gamma}(\gamma_i))}{\sum_{j \in I_\beta \cup I_\gamma} (B_j M_2 + B_j M_1 + C_j)},$$

where $H_i = \min(\frac{\beta_i M_1}{M_3}, \frac{\alpha M_1}{2k(M_4 - M_2 + 1)})$. So in this case $\{x_n\}$ is bounded below by a constant. Now suppose that $I_D \cup I_E \subset I_\delta \cup I_\epsilon$, $p > 0$, and $I_C \neq \emptyset$ then we have

$$y_n = \frac{p + \sum_{i=1}^k \delta_i x_{n-i} + \sum_{i=1}^k \epsilon_i y_{n-i}}{q + \sum_{j=1}^k D_j x_{n-j} + \sum_{j=1}^k E_j y_{n-j}} \geq \frac{\frac{p}{2} + \sum_{i \in I_\delta} S_i y_{n-i} + \sum_{i \in I_\epsilon} \epsilon_i y_{n-i}}{q + \sum_{j \in I_\delta \cup I_\epsilon} (D_j M_2 + D_j M_1 y_{n-j} + E_j y_{n-j})}$$
$$\geq \frac{\min(\frac{p}{2}, \min_{i \in I_\delta}(S_i), \min_{i \in I_\epsilon}(\epsilon_i))}{\sum_{j \in I_\delta \cup I_\epsilon} (q + D_j M_2 + D_j M_1 + E_j)},$$

where $S_i = \min(\frac{\delta_i M_1}{M_3}, \frac{p M_1}{2k(M_4 - M_2 + 1)})$. So in this case $\{y_n\}$ is bounded below by a constant. Now we have shown that there exists $A_2 > 0$, so that $\{x_n\}$ is bounded below by A_2 and $I_B \neq \emptyset$ or $\{y_n\}$ is bounded below by A_2 and $I_C \neq \emptyset$. We use this fact to show the following.

$$x_n = \frac{\alpha + \sum_{i=1}^k \beta_i x_{n-i} + \sum_{i=1}^k \gamma_i y_{n-i}}{\sum_{j=1}^k B_j x_{n-j} + \sum_{j=1}^k C_j y_{n-j}}$$
$$\leq 2 \frac{\alpha + \sum_{i=1}^k \beta_i x_{n-i} + \sum_{i=1}^k \gamma_i (\frac{M_3 x_{n-i} + M_4 - M_2}{M_1})}{\min_+(\sum_{j=1}^k B_j, \sum_{j=1}^k C_j) A_2 + \sum_{j=1}^k B_j x_{n-j} + \sum_{j=1}^k C_j y_{n-j}}$$
$$\leq 2 \frac{\alpha + \sum_{i=1}^k \beta_i x_{n-i} + \sum_{i=1}^k \gamma_i (\frac{M_3 x_{n-i} + M_4 - M_2}{M_1})}{\min_+(\sum_{j=1}^k B_j, \sum_{j=1}^k C_j) \frac{A_2}{2} + \sum_{j=1}^k B_j x_{n-j} + \sum_{j=1}^k L_j x_{n-j}},$$

where $L_j = \min(\frac{C_j}{M_1}, \frac{\min_+(\sum_{j=1}^k B_j, \sum_{j=1}^k C_j) A_2}{2k(M_2+1)})$. Thus we see that the sequence $\{x_n\}_{n=1}^\infty$ satisfies the above difference inequality. Moreover using Theorem 1 in [7] we see that there exists $M > 0$ and $N \in \mathbb{N}$ so that given any non-negative initial conditions, we have $x_n \leq M$ for all $n > N$. Since there exists $M_1, M_3 > 0$ and $M_4 > M_2 > 0$ so that $x_n \leq M_1 y_n + M_2 \leq M_3 x_n + M_4$ for all $n \in \mathbb{N}$, we have the full result.

□

For the following theorem we first show that x_n or y_n is bounded below by a constant. Then we use Theorem 1 of [7] to iterate with respect to y_n. The fact that there exists $M_1, M_3 > 0$ and $M_4 \geq M_2 \geq 0$ so that $x_n \leq M_1 y_n + M_2 \leq M_3 x_n + M_4$ for all $n \in \mathbb{N}$ gives us the full result.

Theorem 23. *Suppose that we have a k^{th} order system of two rational difference equations*

$$x_n = \frac{\alpha + \sum_{i=1}^k \beta_i x_{n-i} + \sum_{i=1}^k \gamma_i y_{n-i}}{A + \sum_{j=1}^k B_j x_{n-j} + \sum_{j=1}^k C_j y_{n-j}}, \quad n \in \mathbb{N},$$

$$y_n = \frac{p + \sum_{i=1}^{k} \delta_i x_{n-i} + \sum_{i=1}^{k} \epsilon_i y_{n-i}}{q + \sum_{j=1}^{k} D_j x_{n-j} + \sum_{j=1}^{k} E_j y_{n-j}}, \quad n \in \mathbb{N},$$

with non-negative parameters and non-negative initial conditions. Further assume that there exists $M_1, M_3 > 0$ and $M_4 > M_2 > 0$ so that $x_n \leq M_1 y_n + M_2 \leq M_3 x_n + M_4$ for all $n \in \mathbb{N}$, and suppose that $q = 0$ and one of the following holds

(i) $I_B \cup I_C \subset I_\beta \cup I_\gamma$, $\alpha > 0$, and $I_D \neq \emptyset$
(ii) $I_D \cup I_E \subset I_\delta \cup I_\epsilon$, $p > 0$, and $I_E \neq \emptyset$

and there exists a positive integer η, such that for every sequence $\{c_m\}_{m=1}^{\infty}$ with $c_m \in I_\delta \cup I_\epsilon$ for $m = 1, 2, \ldots$ there exists positive integers, N_1,$N_2 \leq \eta$, such that $\sum_{m=N_1}^{N_2} c_m \in I_D \cup I_E$. Then there exists $M > 0$ and $N \in \mathbb{N}$ so that given any non-negative initial conditions, we have $x_n, y_n \leq M$ for all $n > N$.

Proof. Suppose that $I_B \cup I_C \subset I_\beta \cup I_\gamma$, $\alpha > 0$, and $I_D \neq \emptyset$ then we have

$$x_n = \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{n-i} + \sum_{i=1}^{k} \gamma_i y_{n-i}}{A + \sum_{j=1}^{k} B_j x_{n-j} + \sum_{j=1}^{k} C_j y_{n-j}} \geq \frac{\frac{\alpha}{2} + \sum_{i \in I_\beta} H_i y_{n-i} + \sum_{i \in I_\gamma} \gamma_i y_{n-i}}{A + \sum_{j \in I_\beta \cup I_\gamma} (B_j M_2 + B_j M_1 y_{n-j} + C_j y_{n-j})}$$
$$\geq \frac{\min(\frac{\alpha}{2}, \min_{i \in I_\beta}(H_i), \min_{i \in I_\gamma}(\gamma_i))}{\sum_{j \in I_\beta \cup I_\gamma} (A + B_j M_2 + B_j M_1 + C_j)},$$

where $H_i = \min(\frac{\beta_i M_1}{M_3}, \frac{\alpha M_1}{2k(M_4 - M_2 + 1)})$. So in this case $\{x_n\}$ is bounded below by a constant. Now suppose that $I_D \cup I_E \subset I_\delta \cup I_\epsilon$, $p > 0$, and $I_E \neq \emptyset$ then we have

$$y_n = \frac{p + \sum_{i=1}^{k} \delta_i x_{n-i} + \sum_{i=1}^{k} \epsilon_i y_{n-i}}{\sum_{j=1}^{k} D_j x_{n-j} + \sum_{j=1}^{k} E_j y_{n-j}} \geq \frac{\frac{p}{2} + \sum_{i \in I_\delta} S_i y_{n-i} + \sum_{i \in I_\epsilon} \epsilon_i y_{n-i}}{\sum_{j \in I_\delta \cup I_\epsilon} (D_j M_2 + D_j M_1 y_{n-j} + E_j y_{n-j})}$$
$$\geq \frac{\min(\frac{p}{2}, \min_{i \in I_\delta}(S_i), \min_{i \in I_\epsilon}(\epsilon_i))}{\sum_{j \in I_\delta \cup I_\epsilon} (D_j M_2 + D_j M_1 + E_j)},$$

where $S_i = \min(\frac{\delta_i M_1}{M_3}, \frac{p M_1}{2k(M_4 - M_2 + 1)})$. So in this case $\{y_n\}$ is bounded below by a constant. Now we have shown that there exists $q_2 > 0$, so that $\{x_n\}$ is bounded below by q_2 and $I_D \neq \emptyset$ or $\{y_n\}$ is bounded below by q_2 and $I_E \neq \emptyset$. We use this fact to show the following.

$$y_n = \frac{p + \sum_{i=1}^{k} \delta_i x_{n-i} + \sum_{i=1}^{k} \epsilon_i y_{n-i}}{\sum_{j=1}^{k} D_j x_{n-j} + \sum_{j=1}^{k} E_j y_{n-j}}$$
$$\leq 2 \frac{p + \sum_{i=1}^{k} \delta_i M_1 y_{n-i} + \sum_{i=1}^{k} \epsilon_i y_{n-i} + \sum_{i=1}^{k} \delta_i M_2}{\min_+(\sum_{j=1}^{k} D_j, \sum_{j=1}^{k} E_j) q_2 + \sum_{j=1}^{k} D_j x_{n-j} + \sum_{j=1}^{k} E_j y_{n-j}}$$
$$\leq 2 \frac{p + \sum_{i=1}^{k} \delta_i M_1 y_{n-i} + \sum_{i=1}^{k} \epsilon_i y_{n-i} + \sum_{i=1}^{k} \delta_i M_2}{\min_+(\sum_{j=1}^{k} D_j, \sum_{j=1}^{k} E_j) \frac{q_2}{2} + \sum_{j=1}^{k} R_j y_{n-j} + \sum_{j=1}^{k} E_j y_{n-j}},$$

where $R_j = \min(\frac{D_j M_1}{M_3}, \frac{\min_+(\sum_{j=1}^{k} D_j, \sum_{j=1}^{k} E_j) q_2 M_1}{2k(M_4 - M_2 + 1)})$. Thus we see that the sequence $\{y_n\}_{n=1}^{\infty}$ satisfies the above difference inequality. Again using Theorem 1 in [7] we see that there exists $M > 0$ and $N \in \mathbb{N}$ so that given any non-negative initial conditions, we have $y_n \leq M$ for all $n > N$. Since there exists $M_1, M_3 > 0$ and $M_4 > M_2 > 0$ so that

$x_n \leq M_1 y_n + M_2 \leq M_3 x_n + M_4$ for all $n \in \mathbb{N}$, we have the full result.

□

6. Conditions of Comparability

Since the comparisons presented above provide us with such useful tools to study boundedness we devote this section to determining whether a system of two equations satisfies any of the necessary inequalities.

Theorem 24. *Suppose that we have a k^{th} order system of two rational difference equations*

$$x_n = \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{n-i} + \sum_{i=1}^{k} \gamma_i y_{n-i}}{A + \sum_{j=1}^{k} B_j x_{n-j} + \sum_{j=1}^{k} C_j y_{n-j}}, \quad n \in \mathbb{N},$$

$$y_n = \frac{p + \sum_{i=1}^{k} \delta_i x_{n-i} + \sum_{i=1}^{k} \epsilon_i y_{n-i}}{q + \sum_{j=1}^{k} D_j x_{n-j} + \sum_{j=1}^{k} E_j y_{n-j}}, \quad n \in \mathbb{N},$$

with non-negative parameters and non-negative initial conditions. Further suppose that $I_\delta \subset I_\beta$, $I_B \subset I_D$, $I_\epsilon \subset I_\gamma$, $I_C \subset I_E$. Also assume that whenever $A > 0$, then $q > 0$, and whenever $p > 0$, then $\alpha > 0$. Then there exists $M_1 > 0$ so that $y_n \leq M_1 x_n$ for all $n \in \mathbb{N}$.

Proof. First notice that it suffices to show that eventually for $n \geq N$ there exists $M > 0$ so that $y_n \leq M x_n$. This is since we may take $M_1 = \max(M, \max_{n\in L} \frac{y_n}{x_n})$, where $L = \{n \in \mathbb{N} | x_n \neq 0 \text{ and } n < N\}$. Notice that since $I_\delta \subset I_\beta$, $I_\epsilon \subset I_\gamma$, and whenever $p > 0$, then $\alpha > 0$ we get that whenever $x_n = 0$, then $y_n = 0$. Thus $y_n \leq M_1 x_n$ for all $n \in \mathbb{N}$ with $x_n = 0$. Now let us prove that the inequality eventually holds. In the case where $A > 0$ and $p > 0$ we have

$$y_n = \frac{p + \sum_{i=1}^{k} \delta_i x_{n-i} + \sum_{i=1}^{k} \epsilon_i y_{n-i}}{q + \sum_{j=1}^{k} D_j x_{n-j} + \sum_{j=1}^{k} E_j y_{n-j}} \leq$$

$$\left(\frac{\max(p, \max_{i\in I_\delta}(\delta_i), \max_{i\in I_\epsilon}(\epsilon_i))}{\min(q, \min_{j\in I_D}(D_j), \min_{j\in I_E}(E_j))}\right) \frac{1 + \sum_{i\in I_\delta} x_{n-i} + \sum_{i\in I_\epsilon} y_{n-i}}{1 + \sum_{j\in I_D} x_{n-j} + \sum_{j\in I_E} y_{n-j}} \leq$$

$$\left(\frac{\max(p, \max_{i\in I_\delta}(\delta_i), \max_{i\in I_\epsilon}(\epsilon_i))}{\min(q, \min_{j\in I_D}(D_j), \min_{j\in I_E}(E_j))}\right) \frac{1 + \sum_{i\in I_\beta} x_{n-i} + \sum_{i\in I_\gamma} y_{n-i}}{1 + \sum_{j\in I_B} x_{n-j} + \sum_{j\in I_C} y_{n-j}} \leq$$

$$\left(\frac{(\max(A, \max_{j\in I_B}(B_j), \max_{j\in I_C}(C_j)))(\max(p, \max_{i\in I_\delta}(\delta_i), \max_{i\in I_\epsilon}(\epsilon_i)))}{(\min(\alpha, \min_{i\in I_\beta}(\beta_i), \min_{i\in I_\gamma}(\gamma_i)))(\min(q, \min_{j\in I_D}(D_j), \min_{j\in I_E}(E_j)))}\right) x_n.$$

In the case where $A > 0$, $p = 0$, and $\alpha > 0$ we have

$$y_n = \frac{\sum_{i=1}^{k} \delta_i x_{n-i} + \sum_{i=1}^{k} \epsilon_i y_{n-i}}{q + \sum_{j=1}^{k} D_j x_{n-j} + \sum_{j=1}^{k} E_j y_{n-j}} \leq$$

$$\left(\frac{\max(\max_{i\in I_\delta}(\delta_i), \max_{i\in I_\epsilon}(\epsilon_i))}{\min(q, \min_{j\in I_D}(D_j), \min_{j\in I_E}(E_j))}\right) \frac{\sum_{i\in I_\delta} x_{n-i} + \sum_{i\in I_\epsilon} y_{n-i}}{1 + \sum_{j\in I_D} x_{n-j} + \sum_{j\in I_E} y_{n-j}} \leq$$

$$\left(\frac{\max(\max_{i\in I_\delta}(\delta_i),\max_{i\in I_\epsilon}(\epsilon_i))}{\min(q,\min_{j\in I_D}(D_j),\min_{j\in I_E}(E_j))}\right)\frac{1+\sum_{i\in I_\beta}x_{n-i}+\sum_{i\in I_\gamma}y_{n-i}}{1+\sum_{j\in I_B}x_{n-j}+\sum_{j\in I_C}y_{n-j}}\leq$$
$$\left(\frac{(\max(A,\max_{j\in I_B}(B_j),\max_{j\in I_C}(C_j)))(\max(\max_{i\in I_\delta}(\delta_i),\max_{i\in I_\epsilon}(\epsilon_i)))}{(\min(\alpha,\min_{i\in I_\beta}(\beta_i),\min_{i\in I_\gamma}(\gamma_i)))(\min(q,\min_{j\in I_D}(D_j),\min_{j\in I_E}(E_j)))}\right)x_n.$$

In the case where $A>0$, $p=0$, and $\alpha=0$ we have

$$y_n=\frac{\sum_{i=1}^k\delta_ix_{n-i}+\sum_{i=1}^k\epsilon_iy_{n-i}}{q+\sum_{j=1}^kD_jx_{n-j}+\sum_{j=1}^kE_jy_{n-j}}\leq$$
$$\left(\frac{\max(\max_{i\in I_\delta}(\delta_i),\max_{i\in I_\epsilon}(\epsilon_i))}{\min(q,\min_{j\in I_D}(D_j),\min_{j\in I_E}(E_j))}\right)\frac{\sum_{i\in I_\delta}x_{n-i}+\sum_{i\in I_\epsilon}y_{n-i}}{1+\sum_{j\in I_D}x_{n-j}+\sum_{j\in I_E}y_{n-j}}\leq$$
$$\left(\frac{\max(\max_{i\in I_\delta}(\delta_i),\max_{i\in I_\epsilon}(\epsilon_i))}{\min(q,\min_{j\in I_D}(D_j),\min_{j\in I_E}(E_j))}\right)\frac{\sum_{i\in I_\beta}x_{n-i}+\sum_{i\in I_\gamma}y_{n-i}}{1+\sum_{j\in I_B}x_{n-j}+\sum_{j\in I_C}y_{n-j}}\leq$$
$$\left(\frac{(\max(A,\max_{j\in I_B}(B_j),\max_{j\in I_C}(C_j)))(\max(\max_{i\in I_\delta}(\delta_i),\max_{i\in I_\epsilon}(\epsilon_i)))}{(\min(\min_{i\in I_\beta}(\beta_i),\min_{i\in I_\gamma}(\gamma_i)))(\min(q,\min_{j\in I_D}(D_j),\min_{j\in I_E}(E_j)))}\right)x_n.$$

In the case where $A=0$, $q>0$, and $p>0$ we have

$$y_n=\frac{p+\sum_{i=1}^k\delta_ix_{n-i}+\sum_{i=1}^k\epsilon_iy_{n-i}}{q+\sum_{j=1}^kD_jx_{n-j}+\sum_{j=1}^kE_jy_{n-j}}\leq$$
$$\left(\frac{\max(p,\max_{i\in I_\delta}(\delta_i),\max_{i\in I_\epsilon}(\epsilon_i))}{\min(q,\min_{j\in I_D}(D_j),\min_{j\in I_E}(E_j))}\right)\frac{1+\sum_{i\in I_\delta}x_{n-i}+\sum_{i\in I_\epsilon}y_{n-i}}{1+\sum_{j\in I_D}x_{n-j}+\sum_{j\in I_E}y_{n-j}}\leq$$
$$\left(\frac{\max(p,\max_{i\in I_\delta}(\delta_i),\max_{i\in I_\epsilon}(\epsilon_i))}{\min(q,\min_{j\in I_D}(D_j),\min_{j\in I_E}(E_j))}\right)\frac{1+\sum_{i\in I_\beta}x_{n-i}+\sum_{i\in I_\gamma}y_{n-i}}{\sum_{j\in I_B}x_{n-j}+\sum_{j\in I_C}y_{n-j}}\leq$$
$$\left(\frac{(\max(\max_{j\in I_B}(B_j),\max_{j\in I_C}(C_j)))(\max(p,\max_{i\in I_\delta}(\delta_i),\max_{i\in I_\epsilon}(\epsilon_i)))}{(\min(\alpha,\min_{i\in I_\beta}(\beta_i),\min_{i\in I_\gamma}(\gamma_i)))(\min(q,\min_{j\in I_D}(D_j),\min_{j\in I_E}(E_j)))}\right)x_n.$$

In the case where $A=0$, $q>0$, $p=0$, and $\alpha=0$ we have

$$y_n=\frac{\sum_{i=1}^k\delta_ix_{n-i}+\sum_{i=1}^k\epsilon_iy_{n-i}}{q+\sum_{j=1}^kD_jx_{n-j}+\sum_{j=1}^kE_jy_{n-j}}\leq$$
$$\left(\frac{\max(\max_{i\in I_\delta}(\delta_i),\max_{i\in I_\epsilon}(\epsilon_i))}{\min(q,\min_{j\in I_D}(D_j),\min_{j\in I_E}(E_j))}\right)\frac{\sum_{i\in I_\delta}x_{n-i}+\sum_{i\in I_\epsilon}y_{n-i}}{1+\sum_{j\in I_D}x_{n-j}+\sum_{j\in I_E}y_{n-j}}\leq$$
$$\left(\frac{\max(\max_{i\in I_\delta}(\delta_i),\max_{i\in I_\epsilon}(\epsilon_i))}{\min(q,\min_{j\in I_D}(D_j),\min_{j\in I_E}(E_j))}\right)\frac{\sum_{i\in I_\beta}x_{n-i}+\sum_{i\in I_\gamma}y_{n-i}}{\sum_{j\in I_B}x_{n-j}+\sum_{j\in I_C}y_{n-j}}\leq$$
$$\left(\frac{(\max(\max_{j\in I_B}(B_j),\max_{j\in I_C}(C_j)))(\max(\max_{i\in I_\delta}(\delta_i),\max_{i\in I_\epsilon}(\epsilon_i)))}{(\min(\min_{i\in I_\beta}(\beta_i),\min_{i\in I_\gamma}(\gamma_i)))(\min(q,\min_{j\in I_D}(D_j),\min_{j\in I_E}(E_j)))}\right)x_n.$$

In the case where $A=0$, $q=0$, and $p>0$ we have

$$y_n=\frac{p+\sum_{i=1}^k\delta_ix_{n-i}+\sum_{i=1}^k\epsilon_iy_{n-i}}{\sum_{j=1}^kD_jx_{n-j}+\sum_{j=1}^kE_jy_{n-j}}\leq$$

$$\left(\frac{\max(p,\max_{i\in I_\delta}(\delta_i),\max_{i\in I_\epsilon}(\epsilon_i))}{\min(\min_{j\in I_D}(D_j),\min_{j\in I_E}(E_j))}\right)\frac{1+\sum_{i\in I_\delta}x_{n-i}+\sum_{i\in I_\epsilon}y_{n-i}}{\sum_{j\in I_D}x_{n-j}+\sum_{j\in I_E}y_{n-j}}\leq$$

$$\left(\frac{\max(p,\max_{i\in I_\delta}(\delta_i),\max_{i\in I_\epsilon}(\epsilon_i))}{\min(\min_{j\in I_D}(D_j),\min_{j\in I_E}(E_j))}\right)\frac{1+\sum_{i\in I_\beta}x_{n-i}+\sum_{i\in I_\gamma}y_{n-i}}{\sum_{j\in I_B}x_{n-j}+\sum_{j\in I_C}y_{n-j}}\leq$$

$$\left(\frac{(\max(\max_{j\in I_B}(B_j),\max_{j\in I_C}(C_j)))(\max(p,\max_{i\in I_\delta}(\delta_i),\max_{i\in I_\epsilon}(\epsilon_i)))}{(\min(\alpha,\min_{i\in I_\beta}(\beta_i),\min_{i\in I_\gamma}(\gamma_i)))(\min(\min_{j\in I_D}(D_j),\min_{j\in I_E}(E_j)))}\right)x_n.$$

In the case where $A=0$, $q=0$, $p=0$, and $\alpha=0$ we have

$$y_n=\frac{\sum_{i=1}^k\delta_i x_{n-i}+\sum_{i=1}^k\epsilon_i y_{n-i}}{\sum_{j=1}^k D_j x_{n-j}+\sum_{j=1}^k E_j y_{n-j}}\leq$$

$$\left(\frac{\max(\max_{i\in I_\delta}(\delta_i),\max_{i\in I_\epsilon}(\epsilon_i))}{\min(\min_{j\in I_D}(D_j),\min_{j\in I_E}(E_j))}\right)\frac{\sum_{i\in I_\delta}x_{n-i}+\sum_{i\in I_\epsilon}y_{n-i}}{\sum_{j\in I_D}x_{n-j}+\sum_{j\in I_E}y_{n-j}}\leq$$

$$\left(\frac{\max(\max_{i\in I_\delta}(\delta_i),\max_{i\in I_\epsilon}(\epsilon_i))}{\min(\min_{j\in I_D}(D_j),\min_{j\in I_E}(E_j))}\right)\frac{\sum_{i\in I_\beta}x_{n-i}+\sum_{i\in I_\gamma}y_{n-i}}{\sum_{j\in I_B}x_{n-j}+\sum_{j\in I_C}y_{n-j}}\leq$$

$$\left(\frac{(\max(\max_{j\in I_B}(B_j),\max_{j\in I_C}(C_j)))(\max(\max_{i\in I_\delta}(\delta_i),\max_{i\in I_\epsilon}(\epsilon_i)))}{(\min(\min_{i\in I_\beta}(\beta_i),\min_{i\in I_\gamma}(\gamma_i)))(\min(\min_{j\in I_D}(D_j),\min_{j\in I_E}(E_j)))}\right)x_n.$$

In the case where $A=0$, $q>0$, $p=0$, and $\alpha>0$ we have

$$y_n=\frac{\sum_{i=1}^k\delta_i x_{n-i}+\sum_{i=1}^k\epsilon_i y_{n-i}}{q+\sum_{j=1}^k D_j x_{n-j}+\sum_{j=1}^k E_j y_{n-j}}\leq$$

$$\left(\frac{\max(\max_{i\in I_\delta}(\delta_i),\max_{i\in I_\epsilon}(\epsilon_i))}{\min(q,\min_{j\in I_D}(D_j),\min_{j\in I_E}(E_j))}\right)\frac{\sum_{i\in I_\delta}x_{n-i}+\sum_{i\in I_\epsilon}y_{n-i}}{1+\sum_{j\in I_D}x_{n-j}+\sum_{j\in I_E}y_{n-j}}\leq$$

$$\left(\frac{\max(\max_{i\in I_\delta}(\delta_i),\max_{i\in I_\epsilon}(\epsilon_i))}{\min(q,\min_{j\in I_D}(D_j),\min_{j\in I_E}(E_j))}\right)\frac{1+\sum_{i\in I_\beta}x_{n-i}+\sum_{i\in I_\gamma}y_{n-i}}{\sum_{j\in I_B}x_{n-j}+\sum_{j\in I_C}y_{n-j}}\leq$$

$$\left(\frac{(\max(\max_{j\in I_B}(B_j),\max_{j\in I_C}(C_j)))(\max(\max_{i\in I_\delta}(\delta_i),\max_{i\in I_\epsilon}(\epsilon_i)))}{(\min(\alpha,\min_{i\in I_\beta}(\beta_i),\min_{i\in I_\gamma}(\gamma_i)))(\min(q,\min_{j\in I_D}(D_j),\min_{j\in I_E}(E_j)))}\right)x_n.$$

In the case where $A=0$, $q=0$, $p=0$, and $\alpha>0$ we have

$$y_n=\frac{\sum_{i=1}^k\delta_i x_{n-i}+\sum_{i=1}^k\epsilon_i y_{n-i}}{\sum_{j=1}^k D_j x_{n-j}+\sum_{j=1}^k E_j y_{n-j}}\leq$$

$$\left(\frac{\max(\max_{i\in I_\delta}(\delta_i),\max_{i\in I_\epsilon}(\epsilon_i))}{\min(\min_{j\in I_D}(D_j),\min_{j\in I_E}(E_j))}\right)\frac{\sum_{i\in I_\delta}x_{n-i}+\sum_{i\in I_\epsilon}y_{n-i}}{\sum_{j\in I_D}x_{n-j}+\sum_{j\in I_E}y_{n-j}}\leq$$

$$\left(\frac{\max(\max_{i\in I_\delta}(\delta_i),\max_{i\in I_\epsilon}(\epsilon_i))}{\min(\min_{j\in I_D}(D_j),\min_{j\in I_E}(E_j))}\right)\frac{1+\sum_{i\in I_\beta}x_{n-i}+\sum_{i\in I_\gamma}y_{n-i}}{\sum_{j\in I_B}x_{n-j}+\sum_{j\in I_C}y_{n-j}}\leq$$

$$\left(\frac{(\max(\max_{j\in I_B}(B_j),\max_{j\in I_C}(C_j)))(\max(\max_{i\in I_\delta}(\delta_i),\max_{i\in I_\epsilon}(\epsilon_i)))}{(\min(\alpha,\min_{i\in I_\beta}(\beta_i),\min_{i\in I_\gamma}(\gamma_i)))(\min(\min_{j\in I_D}(D_j),\min_{j\in I_E}(E_j)))}\right)x_n.$$

$\square$

Theorem 25. *Suppose that we have a k^{th} order system of two rational difference equations*

$$x_n = \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{n-i} + \sum_{i=1}^{k} \gamma_i y_{n-i}}{A + \sum_{j=1}^{k} B_j x_{n-j} + \sum_{j=1}^{k} C_j y_{n-j}}, \quad n \in \mathbb{N},$$

$$y_n = \frac{p + \sum_{i=1}^{k} \delta_i x_{n-i} + \sum_{i=1}^{k} \epsilon_i y_{n-i}}{q + \sum_{j=1}^{k} D_j x_{n-j} + \sum_{j=1}^{k} E_j y_{n-j}}, \quad n \in \mathbb{N},$$

with non-negative parameters and non-negative initial conditions. Further suppose that $I_\beta = I_\delta$, $I_B = I_D$, $I_\gamma = I_\epsilon$, $I_C = I_E$, $\alpha > 0$ if and only if $p > 0$, and $A > 0$ if and only if $q > 0$. Then there exists constants $M_1, M_2 > 0$ so that $M_1 y_n \leq x_n \leq M_2 y_n$ for all $n \in \mathbb{N}$.

Proof. First notice that Theorem 24 applies to this system. This gives us $M_3 > 0$ so that $y_n \leq M_3 x_n$ for all $n \in \mathbb{N}$. Moreover after a very simple change of variables Theorem 24 applies again. The change of variables we refer to here comes from renaming x_n as y_n, β_i as ϵ_i, B_i as E_i, γ_i as δ_i, C_i as D_i, α as p, A as q, and vice versa. This gives us $M_2 > 0$ so that $x_n \leq M_2 y_n$ for all $n \in \mathbb{N}$. Choose $M_1 = \frac{1}{M_3}$ and we get that there exists constants $M_1, M_2 > 0$ so that $M_1 y_n \leq x_n \leq M_2 y_n$ for all $n \in \mathbb{N}$. □

Theorem 26. *Suppose that we have a k^{th} order system of two rational difference equations*

$$x_n = \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{n-i} + \sum_{i=1}^{k} \gamma_i y_{n-i}}{A + \sum_{j=1}^{k} B_j x_{n-j} + \sum_{j=1}^{k} C_j y_{n-j}}, \quad n \in \mathbb{N},$$

$$y_n = \frac{p + \sum_{i=1}^{k} \delta_i x_{n-i} + \sum_{i=1}^{k} \epsilon_i y_{n-i}}{q + \sum_{j=1}^{k} D_j x_{n-j} + \sum_{j=1}^{k} E_j y_{n-j}}, \quad n \in \mathbb{N},$$

with non-negative parameters and non-negative initial conditions. Further suppose that $I_\delta \subset I_\beta \cup I_B$, $I_B \subset I_D$, $I_\epsilon \subset I_\gamma \cup I_C$, $I_C \subset I_E$. Also assume that whenever $A > 0$, then $q > 0$, and whenever $p > 0$, then $\alpha > 0$ or $A > 0$. Then there exists $M_1 > 0$ and $M_2 \geq 0$ so that $y_n \leq M_1 x_n + M_2$ for all $n \in \mathbb{N}$.

Proof. First notice that it suffices to show that eventually for $n \geq N$ there exists $M_3 > 0$ and $M_4 \geq 0$ so that $y_n \leq M_3 x_n + M_4$. This is since we may take $M_1 = M_3$ and $M_2 = \max(M_4, \max_{n<N} y_n)$ and so there exists $M_1 > 0$ and $M_2 \geq 0$ so that $y_n \leq M_1 x_n + M_2$ for all $n \in \mathbb{N}$. In the case where $A > 0$ and $p > 0$ we have

$$y_n = \frac{p + \sum_{i=1}^{k} \delta_i x_{n-i} + \sum_{i=1}^{k} \epsilon_i y_{n-i}}{q + \sum_{j=1}^{k} D_j x_{n-j} + \sum_{j=1}^{k} E_j y_{n-j}} \leq$$

$$\left(\frac{\max(p, \max_{i \in I_\delta}(\delta_i), \max_{i \in I_\epsilon}(\epsilon_i))}{\min(q, \min_{j \in I_D}(D_j), \min_{j \in I_E}(E_j))} \right) \frac{1 + \sum_{i \in I_\delta} x_{n-i} + \sum_{i \in I_\epsilon} y_{n-i}}{1 + \sum_{j \in I_D} x_{n-j} + \sum_{j \in I_E} y_{n-j}} \leq$$

$$\left(\frac{\max(p, \max_{i \in I_\delta}(\delta_i), \max_{i \in I_\epsilon}(\epsilon_i))}{\min(q, \min_{j \in I_D}(D_j), \min_{j \in I_E}(E_j))} \right) \frac{1 + \sum_{i \in I_\beta \cup I_B} x_{n-i} + \sum_{i \in I_\gamma \cup I_C} y_{n-i}}{1 + \sum_{j \in I_B} x_{n-j} + \sum_{j \in I_C} y_{n-j}} \leq$$

$$M_1\left(\frac{\alpha+A+\sum_{i=1}^{k}(\beta_i+B_i)x_{n-i}+\sum_{i=1}^{k}(\gamma_i+C_i)y_{n-i}}{A+\sum_{j=1}^{k}B_jx_{n-j}+\sum_{j=1}^{k}C_jy_{n-j}}\right)=M_1(x_n+1),$$

where

$$M_1=\frac{(\max(A,\max_{j\in I_B}(B_j),\max_{j\in I_C}(C_j)))(\max(p,\max_{i\in I_\delta}(\delta_i),\max_{i\in I_\epsilon}(\epsilon_i)))}{(\min(\alpha+A,\min_{i\in I_\beta\cup I_B}(\beta_i+B_i),\min_{i\in I_\gamma\cup I_C}(\gamma_i+C_i)))(\min(q,\min_{j\in I_D}(D_j),\min_{j\in I_E}(E_j)))}.$$

In the case where $A>0$ and $p=0$ we have

$$y_n=\frac{\sum_{i=1}^{k}\delta_ix_{n-i}+\sum_{i=1}^{k}\epsilon_iy_{n-i}}{q+\sum_{j=1}^{k}D_jx_{n-j}+\sum_{j=1}^{k}E_jy_{n-j}}\leq$$

$$\left(\frac{\max(\max_{i\in I_\delta}(\delta_i),\max_{i\in I_\epsilon}(\epsilon_i))}{\min(q,\min_{j\in I_D}(D_j),\min_{j\in I_E}(E_j))}\right)\frac{\sum_{i\in I_\delta}x_{n-i}+\sum_{i\in I_\epsilon}y_{n-i}}{1+\sum_{j\in I_D}x_{n-j}+\sum_{j\in I_E}y_{n-j}}\leq$$

$$\left(\frac{\max(\max_{i\in I_\delta}(\delta_i),\max_{i\in I_\epsilon}(\epsilon_i))}{\min(q,\min_{j\in I_D}(D_j),\min_{j\in I_E}(E_j))}\right)\left(\frac{1+\sum_{i\in I_\beta\cup I_B}x_{n-i}+\sum_{i\in I_\gamma\cup I_C}y_{n-i}}{1+\sum_{j\in I_B}x_{n-j}+\sum_{j\in I_C}y_{n-j}}\right)\leq M_1(x_n+1),$$

where

$$M_1=\frac{(\max(A,\max_{j\in I_B}(B_j),\max_{j\in I_C}(C_j)))(\max(\max_{i\in I_\delta}(\delta_i),\max_{i\in I_\epsilon}(\epsilon_i)))}{(\min(\alpha+A,\min_{i\in I_\beta\cup I_B}(\beta_i+B_i),\min_{i\in I_\gamma\cup I_C}(\gamma_i+C_i)))(\min(q,\min_{j\in I_D}(D_j),\min_{j\in I_E}(E_j)))}.$$

In the case where $A=0$, $q>0$, and $p>0$ we have

$$y_n=\frac{p+\sum_{i=1}^{k}\delta_ix_{n-i}+\sum_{i=1}^{k}\epsilon_iy_{n-i}}{q+\sum_{j=1}^{k}D_jx_{n-j}+\sum_{j=1}^{k}E_jy_{n-j}}\leq$$

$$\left(\frac{\max(p,\max_{i\in I_\delta}(\delta_i),\max_{i\in I_\epsilon}(\epsilon_i))}{\min(q,\min_{j\in I_D}(D_j),\min_{j\in I_E}(E_j))}\right)\frac{1+\sum_{i\in I_\delta}x_{n-i}+\sum_{i\in I_\epsilon}y_{n-i}}{1+\sum_{j\in I_D}x_{n-j}+\sum_{j\in I_E}y_{n-j}}\leq$$

$$\left(\frac{\max(p,\max_{i\in I_\delta}(\delta_i),\max_{i\in I_\epsilon}(\epsilon_i))}{\min(q,\min_{j\in I_D}(D_j),\min_{j\in I_E}(E_j))}\right)\left(\frac{1+\sum_{i\in I_\beta\cup I_B}x_{n-i}+\sum_{i\in I_\gamma\cup I_C}y_{n-i}}{\sum_{j\in I_B}x_{n-j}+\sum_{j\in I_C}y_{n-j}}\right)\leq M_1(x_n+1),$$

where

$$M_1=\frac{(\max(\max_{j\in I_B}(B_j),\max_{j\in I_C}(C_j)))(\max(p,\max_{i\in I_\delta}(\delta_i),\max_{i\in I_\epsilon}(\epsilon_i)))}{(\min(\alpha+A,\min_{i\in I_\beta\cup I_B}(\beta_i+B_i),\min_{i\in I_\gamma\cup I_C}(\gamma_i+C_i)))(\min(q,\min_{j\in I_D}(D_j),\min_{j\in I_E}(E_j)))}.$$

In the case where $A=0$, $q>0$, $p=0$, and $\alpha=0$ we have

$$y_n=\frac{\sum_{i=1}^{k}\delta_ix_{n-i}+\sum_{i=1}^{k}\epsilon_iy_{n-i}}{q+\sum_{j=1}^{k}D_jx_{n-j}+\sum_{j=1}^{k}E_jy_{n-j}}\leq$$

$$\left(\frac{\max(\max_{i\in I_\delta}(\delta_i),\max_{i\in I_\epsilon}(\epsilon_i))}{\min(q,\min_{j\in I_D}(D_j),\min_{j\in I_E}(E_j))}\right)\frac{\sum_{i\in I_\delta}x_{n-i}+\sum_{i\in I_\epsilon}y_{n-i}}{1+\sum_{j\in I_D}x_{n-j}+\sum_{j\in I_E}y_{n-j}}\leq$$

$$\left(\frac{\max(\max_{i\in I_\delta}(\delta_i),\max_{i\in I_\epsilon}(\epsilon_i))}{\min(q,\min_{j\in I_D}(D_j),\min_{j\in I_E}(E_j))}\right)\left(\frac{\sum_{i\in I_\beta\cup I_B}x_{n-i}+\sum_{i\in I_\gamma\cup I_C}y_{n-i}}{\sum_{j\in I_B}x_{n-j}+\sum_{j\in I_C}y_{n-j}}\right)\leq M_1(x_n+1),$$

where

$$M_1 = \frac{(\max(\max_{j\in I_B}(B_j), \max_{j\in I_C}(C_j)))(\max(\max_{i\in I_\delta}(\delta_i), \max_{i\in I_\epsilon}(\epsilon_i)))}{(\min(\min_{i\in I_\beta\cup I_B}(\beta_i+B_i), \min_{i\in I_\gamma\cup I_C}(\gamma_i+C_i)))(\min(q, \min_{j\in I_D}(D_j), \min_{j\in I_E}(E_j)))}.$$

In the case where $A = 0$, $q = 0$, and $p > 0$ we have

$$y_n = \frac{p + \sum_{i=1}^k \delta_i x_{n-i} + \sum_{i=1}^k \epsilon_i y_{n-i}}{\sum_{j=1}^k D_j x_{n-j} + \sum_{j=1}^k E_j y_{n-j}} \le$$

$$\left(\frac{\max(p, \max_{i\in I_\delta}(\delta_i), \max_{i\in I_\epsilon}(\epsilon_i))}{\min(\min_{j\in I_D}(D_j), \min_{j\in I_E}(E_j))}\right) \frac{1 + \sum_{i\in I_\delta} x_{n-i} + \sum_{i\in I_\epsilon} y_{n-i}}{\sum_{j\in I_D} x_{n-j} + \sum_{j\in I_E} y_{n-j}} \le$$

$$\left(\frac{\max(p, \max_{i\in I_\delta}(\delta_i), \max_{i\in I_\epsilon}(\epsilon_i))}{\min(\min_{j\in I_D}(D_j), \min_{j\in I_E}(E_j))}\right)\left(\frac{1 + \sum_{i\in I_\beta\cup I_B} x_{n-i} + \sum_{i\in I_\gamma\cup I_C} y_{n-i}}{\sum_{j\in I_B} x_{n-j} + \sum_{j\in I_C} y_{n-j}}\right) \le M_1(x_n+1),$$

where

$$M_1 = \frac{(\max(\max_{j\in I_B}(B_j), \max_{j\in I_C}(C_j)))(\max(p, \max_{i\in I_\delta}(\delta_i), \max_{i\in I_\epsilon}(\epsilon_i)))}{(\min(\alpha + A, \min_{i\in I_\beta\cup I_B}(\beta_i+B_i), \min_{i\in I_\gamma\cup I_C}(\gamma_i+C_i)))(\min(\min_{j\in I_D}(D_j), \min_{j\in I_E}(E_j)))}.$$

In the case where $A = 0$, $q = 0$, $p = 0$, and $\alpha = 0$ we have

$$y_n = \frac{\sum_{i=1}^k \delta_i x_{n-i} + \sum_{i=1}^k \epsilon_i y_{n-i}}{\sum_{j=1}^k D_j x_{n-j} + \sum_{j=1}^k E_j y_{n-j}} \le$$

$$\left(\frac{\max(\max_{i\in I_\delta}(\delta_i), \max_{i\in I_\epsilon}(\epsilon_i))}{\min(\min_{j\in I_D}(D_j), \min_{j\in I_E}(E_j))}\right) \frac{\sum_{i\in I_\delta} x_{n-i} + \sum_{i\in I_\epsilon} y_{n-i}}{\sum_{j\in I_D} x_{n-j} + \sum_{j\in I_E} y_{n-j}} \le$$

$$\left(\frac{\max(\max_{i\in I_\delta}(\delta_i), \max_{i\in I_\epsilon}(\epsilon_i))}{\min(\min_{j\in I_D}(D_j), \min_{j\in I_E}(E_j))}\right)\left(\frac{\sum_{i\in I_\beta\cup I_B} x_{n-i} + \sum_{i\in I_\gamma\cup I_C} y_{n-i}}{\sum_{j\in I_B} x_{n-j} + \sum_{j\in I_C} y_{n-j}}\right) \le M_1(x_n+1),$$

where

$$M_1 = \frac{(\max(\max_{j\in I_B}(B_j), \max_{j\in I_C}(C_j)))(\max(\max_{i\in I_\delta}(\delta_i), \max_{i\in I_\epsilon}(\epsilon_i)))}{(\min(\min_{i\in I_\beta\cup I_B}(\beta_i+B_i), \min_{i\in I_\gamma\cup I_C}(\gamma_i+C_i)))(\min(\min_{j\in I_D}(D_j), \min_{j\in I_E}(E_j)))}.$$

In the case where $A = 0$, $q > 0$, $p = 0$, and $\alpha > 0$ we have

$$y_n = \frac{\sum_{i=1}^k \delta_i x_{n-i} + \sum_{i=1}^k \epsilon_i y_{n-i}}{q + \sum_{j=1}^k D_j x_{n-j} + \sum_{j=1}^k E_j y_{n-j}} \le$$

$$\left(\frac{\max(\max_{i\in I_\delta}(\delta_i), \max_{i\in I_\epsilon}(\epsilon_i))}{\min(q, \min_{j\in I_D}(D_j), \min_{j\in I_E}(E_j))}\right) \frac{\sum_{i\in I_\delta} x_{n-i} + \sum_{i\in I_\epsilon} y_{n-i}}{1 + \sum_{j\in I_D} x_{n-j} + \sum_{j\in I_E} y_{n-j}} \le$$

$$\left(\frac{\max(\max_{i\in I_\delta}(\delta_i), \max_{i\in I_\epsilon}(\epsilon_i))}{\min(q, \min_{j\in I_D}(D_j), \min_{j\in I_E}(E_j))}\right)\left(\frac{1 + \sum_{i\in I_\beta\cup I_B} x_{n-i} + \sum_{i\in I_\gamma\cup I_C} y_{n-i}}{\sum_{j\in I_B} x_{n-j} + \sum_{j\in I_C} y_{n-j}}\right) \le M_1(x_n+1),$$

where

$$M_1 = \frac{(\max(\max_{j\in I_B}(B_j), \max_{j\in I_C}(C_j)))(\max(\max_{i\in I_\delta}(\delta_i), \max_{i\in I_\epsilon}(\epsilon_i)))}{(\min(\alpha + A, \min_{i\in I_\beta\cup I_B}(\beta_i+B_i), \min_{i\in I_\gamma\cup I_C}(\gamma_i+C_i)))(\min(q, \min_{j\in I_D}(D_j), \min_{j\in I_E}(E_j)))}.$$

In the case where $A = 0$, $q = 0$, $p = 0$, and $\alpha > 0$ we have

$$y_n = \frac{\sum_{i=1}^k \delta_i x_{n-i} + \sum_{i=1}^k \epsilon_i y_{n-i}}{\sum_{j=1}^k D_j x_{n-j} + \sum_{j=1}^k E_j y_{n-j}} \leq$$

$$\left(\frac{\max(\max_{i\in I_\delta}(\delta_i), \max_{i\in I_\epsilon}(\epsilon_i))}{\min(\min_{j\in I_D}(D_j), \min_{j\in I_E}(E_j))}\right) \frac{\sum_{i\in I_\delta} x_{n-i} + \sum_{i\in I_\epsilon} y_{n-i}}{\sum_{j\in I_D} x_{n-j} + \sum_{j\in I_E} y_{n-j}} \leq$$

$$\left(\frac{\max(\max_{i\in I_\delta}(\delta_i), \max_{i\in I_\epsilon}(\epsilon_i))}{\min(\min_{j\in I_D}(D_j), \min_{j\in I_E}(E_j))}\right) \left(\frac{1 + \sum_{i\in I_\beta\cup I_B} x_{n-i} + \sum_{i\in I_\gamma\cup I_C} y_{n-i}}{\sum_{j\in I_B} x_{n-j} + \sum_{j\in I_C} y_{n-j}}\right) \leq M_1(x_n+1),$$

where

$$M_1 = \frac{(\max(\max_{j\in I_B}(B_j), \max_{j\in I_C}(C_j)))(\max(\max_{i\in I_\delta}(\delta_i), \max_{i\in I_\epsilon}(\epsilon_i)))}{(\min(\alpha + A, \min_{i\in I_\beta\cup I_B}(\beta_i + B_i), \min_{i\in I_\gamma\cup I_C}(\gamma_i + C_i)))(\min(\min_{j\in I_D}(D_j), \min_{j\in I_E}(E_j)))}.$$

□

Theorem 27. *Suppose that we have a k^{th} order system of two rational difference equations*

$$x_n = \frac{\alpha + \sum_{i=1}^k \beta_i x_{n-i} + \sum_{i=1}^k \gamma_i y_{n-i}}{A + \sum_{j=1}^k B_j x_{n-j} + \sum_{j=1}^k C_j y_{n-j}}, \quad n \in \mathbb{N},$$

$$y_n = \frac{p + \sum_{i=1}^k \delta_i x_{n-i} + \sum_{i=1}^k \epsilon_i y_{n-i}}{q + \sum_{j=1}^k D_j x_{n-j} + \sum_{j=1}^k E_j y_{n-j}}, \quad n \in \mathbb{N},$$

with non-negative parameters and non-negative initial conditions. Further suppose that $I_\delta \subset I_\beta \cup I_B$, $I_\beta \subset I_\delta \cup I_D$, $I_B = I_D$, $I_\epsilon \subset I_\gamma \cup I_C$, $I_\gamma \subset I_\epsilon \cup I_E$ $I_C = I_E$. Also assume that $A > 0$ if and only if $q > 0$, and whenever $p > 0$, then $\alpha > 0$ or $A > 0$, also whenever $\alpha > 0$, then $p > 0$ or $q > 0$. Then there exists $M_1, M_3 > 0$ and $M_4 \geq M_2 \geq 0$ so that $x_n \leq M_1 y_n + M_2 \leq M_3 x_n + M_4$ for all $n \in \mathbb{N}$.

Proof. First notice that Theorem 26 applies to this system. This gives us $M_5 > 0$ and $M_6 \geq 0$ so that $y_n \leq M_5 x_n + M_6$ for all $n \in \mathbb{N}$. Moreover after a very simple change of variables Theorem 26 applies again. The change of variables we refer to here comes from renaming x_n as y_n, β_i as ϵ_i, B_i as E_i, γ_i as δ_i, C_i as D_i, α as p, A as q, and vice versa. This gives us $M_1 > 0$ and $M_2 \geq 0$ so that $x_n \leq M_1 y_n + M_2$ for all $n \in \mathbb{N}$. Choose $M_3 = M_1 M_5$ and $M_4 = M_1 M_6 + M_2$ and we get that there exists $M_1, M_3 > 0$ and $M_4 \geq M_2 \geq 0$ so that $x_n \leq M_1 y_n + M_2 \leq M_3 x_n + M_4$ for all $n \in \mathbb{N}$. □

7. Some Examples

Here we present some examples which demonstrate how the results presented earlier in this article are applied to particular special cases.

Example 1. *Consider the following system of two rational difference equations*

$$x_n = \frac{\alpha + \beta_1 x_{n-1}}{A + C_2 y_{n-2}}, \quad n \in \mathbb{N},$$

$$y_n = \frac{p+\delta_1 x_{n-1}}{q+E_2 y_{n-2}}, \quad n \in \mathbb{N}.$$

We assume positive parameters and non-negative initial conditions. This implies that the solutions x_n and y_n are bounded above by a positive constant.

Proof. We apply Theorem 1. First notice that, by theorem 25, that there exists constants $M_1, M_2 > 0$ so that $M_1 y_n \leq x_n \leq M_2 y_n$ for all $n \in \mathbb{N}$. This is since

$$\begin{gathered} \{1\} = I_\beta = I_\delta \\ \emptyset = I_B = I_D \\ \emptyset = I_\gamma = I_\epsilon \\ \{2\} = I_C = I_E \\ \alpha > 0 \quad \textit{and} \quad p > 0 \\ A > 0 \quad \textit{and} \quad q > 0. \end{gathered}$$

For the final condition, let $\eta = 2$ so that for any sequence $\{c_m\}_{m=1}^{\infty}$ with $c_m \in I_\beta \cup I_\gamma = \{1\}$ for $m = 1, 2, \ldots$ we choose $N_1 = 1$,$N_2 = 2 \leq \eta$ so that $\sum_{m=N_1}^{N_2} 1 \in I_B \cup I_C = \{2\}$. $\square$

Example 2. *Consider the following system of two rational difference equations*

$$x_n = \frac{\beta_2 x_{n-2} + \gamma_1 y_{n-1} + \gamma_2 y_{n-2}}{B_2 x_{n-2} + C_1 y_{n-1}}, \quad n \in \mathbb{N},$$

$$y_n = \frac{p + \delta_2 x_{n-2} + \epsilon_1 y_{n-1} + \epsilon_2 y_{n-2}}{q + D_2 x_{n-2} + E_1 y_{n-1}}, \quad n \in \mathbb{N}.$$

We assume positive parameters and positive initial conditions. This implies that the solutions x_n and y_n are bounded above by a positive constant.

Proof. We apply Theorem 3 case (iii). We will now prove that there exists constants $M_1, M_2 > 0$ so that $M_1 y_n \leq x_n \leq M_2 y_n$ for all $n \in \mathbb{N}$. To show this, we first we show that there exists L so that $y_n \geq L$ for all $n \in \mathbb{N}$. We can choose $L = \frac{\min(p,\delta_2,\epsilon_1)}{\max(q,D_2,E_1)}$, since

$$y_n = \frac{p + \delta_2 x_{n-2} + \epsilon_1 y_{n-1} + \epsilon_2 y_{n-2}}{q + D_2 x_{n-2} + E_1 y_{n-1}} \geq \frac{\min(p,\delta_2,\epsilon_1)}{\max(q,D_2,E_1)}.$$

We may choose $M_2 = \max\left(\left(\frac{\max(1,\beta_2,\gamma_1,\gamma_2)}{\min(\frac{C_1 L}{2},B_2,\frac{C_1}{2})}\right)\left(\frac{\max(q,D_2,E_1)}{\min(p,\delta_2,\epsilon_1,\epsilon_2)}\right), \frac{x_1}{y_1}\right)$ since

$$x_n = \frac{\beta_2 x_{n-2} + \gamma_1 y_{n-1} + \gamma_2 y_{n-2}}{B_2 x_{n-2} + C_1 y_{n-1}} \leq \frac{1 + \beta_2 x_{n-2} + \gamma_1 y_{n-1} + \gamma_2 y_{n-2}}{\frac{C_1 L}{2} + B_2 x_{n-2} + \frac{C_1}{2} y_{n-1}}$$

$$\leq \left(\frac{\max(1,\beta_2,\gamma_1,\gamma_2)}{\min(\frac{C_1 L}{2},B_2,\frac{C_1}{2})}\right)\left(\frac{1 + x_{n-2} + y_{n-1} + y_{n-2}}{1 + x_{n-2} + y_{n-1}}\right)$$

$$\leq \left(\frac{\max(1,\beta_2,\gamma_1,\gamma_2)}{\min(\frac{C_1 L}{2},B_2,\frac{C_1}{2})}\right)\left(\frac{\max(q,D_2,E_1)}{\min(p,\delta_2,\epsilon_1,\epsilon_2)}\right) y_n.$$

We may choose $M_1 = \min\left(\left(\frac{\min(\frac{\gamma_1 L}{2},\beta_2,\frac{\gamma_1}{2},\gamma_2)}{\max(1,B_2,C_1)}\right)\left(\frac{\min(q,D_2,E_1)}{\max(p,\delta_2,\epsilon_1,\epsilon_2)}\right), \frac{x_1}{y_1}\right)$ since

$$x_n = \frac{\beta_2 x_{n-2} + \gamma_1 y_{n-1} + \gamma_2 y_{n-2}}{B_2 x_{n-2} + C_1 y_{n-1}} \geq \frac{\frac{\gamma_1 L}{2} + \beta_2 x_{n-2} + \frac{\gamma_1}{2} y_{n-1} + \gamma_2 y_{n-2}}{1 + B_2 x_{n-2} + C_1 y_{n-1}}$$

$$\geq \left(\frac{\min(\frac{\gamma_1 L}{2}, \beta_2, \frac{\gamma_1}{2}, \gamma_2)}{\max(1, B_2, C_1)}\right)\left(\frac{1 + x_{n-2} + y_{n-1} + y_{n-2}}{1 + x_{n-2} + y_{n-1}}\right)$$

$$\geq \left(\frac{\min(\frac{\gamma_1 L}{2}, \beta_2, \frac{\gamma_1}{2}, \gamma_2)}{\max(1, B_2, C_1)}\right)\left(\frac{\min(q, D_2, E_1)}{\max(p, \delta_2, \epsilon_1, \epsilon_2)}\right) y_n.$$

The conditions specific to case (iii) are satisfied since $p, q > 0$, $\{1,2\} = I_D \cup I_E \subset I_\delta \cup I_\epsilon = \{1,2\}$, and $I_C \neq \emptyset$. For the final condition, let $\eta = 1$ so that for any sequence $\{c_m\}_{m=1}^{\infty}$ with $c_m \in I_\beta \cup I_\gamma = \{1,2\}$ for $m = 1, 2, \dots$ we choose $N_1 = 1$, $N_2 = 1 \leq \eta$, since $c_1 \in \{1,2\}$, so that $\sum_{m=N_1}^{N_2} c_m = c_1 \in I_B \cup I_C = \{1,2\}$. □

Example 3. *Consider the following system of two rational difference equations*

$$x_n = \frac{\alpha + \beta_1 x_{n-1} + \gamma_1 y_{n-1}}{B_2 x_{n-2} + C_1 y_{n-1}}, \quad n \in \mathbb{N},$$

$$y_n = \frac{p + \delta_2 x_{n-2} + \epsilon_1 y_{n-1}}{q}, \quad n \in \mathbb{N}.$$

We assume positive parameters and non-negative initial conditions. This implies that the solution x_n is bounded above by a positive constant.

Proof. We apply by Theorem 10 case (iii). The conditions specific for case (iii) are satisfied since

$$\emptyset = I_D \subset I_\delta$$
$$\emptyset = I_E \subset I_\epsilon$$
$$\{1\} = I_\gamma = I_C$$
$$A = 0 \quad with \quad p, q > 0.$$

For the final condition, let $\eta = 2$ so that for any sequence $\{c_m\}_{m=1}^{\infty}$ with $c_m \in I_\beta = \{1\}$, for $m = 1, 2, \dots$, we choose $N_1 = 1$,$N_2 = 2 \leq \eta$, such that $\sum_{m=N_1}^{N_2} 1 \in I_B = \{2\}$. □

Example 4. *Consider the following system of two rational difference equations*

$$x_n = \frac{\beta_1 x_{n-1} + \beta_2 x_{n-2} + \gamma_2 y_{n-2}}{B_2 x_{n-2} + C_1 y_{n-1}}, \quad n \in \mathbb{N},$$

$$y_n = \frac{p + \delta_1 x_{n-1} + \delta_2 x_{n-2} + \epsilon_2 y_{n-2}}{q + D_2 x_{n-2} + E_1 y_{n-1}}, \quad n \in \mathbb{N}.$$

We assume positive parameters and positive initial conditions. This implies that the solutions x_n and y_n are bounded above by a positive constant.

Proof. We apply Theorem 6. We will now prove that there exists constants $M_1, M_2 > 0$ so that $M_1 y_n \leq x_n \leq M_2 y_n$ for all $n \in \mathbb{N}$. To show this, we first show that there exists L so that $y_n \geq L$ or all $n \in \mathbb{N}$. We now show that we may choose $L = \min\left(\frac{\min(p,\delta_1,\delta_2)}{\max(q+E_1\left(\frac{p}{q}\right),D_2,bE_1)}\right), y_1)$. To show this, we first deduce the following inequality

$$\begin{aligned}
y_n = \frac{p+\delta_1 x_{n-1}+\delta_2 x_{n-2}+\epsilon_2 y_{n-2}}{q+D_2 x_{n-2}+E_1 y_{n-1}} &\leq \frac{p}{q} + \frac{\delta_1 x_{n-1}+\delta_2 x_{n-2}+\epsilon_2 y_{n-2}}{D_2 x_{n-2}+E_1 y_{n-1}} \\
&\leq \frac{p}{q} + \left(\frac{\max(\delta_1,\delta_2,\epsilon_2)}{\min(D_2,E_1)}\right)\left(\frac{x_{n-1}+x_{n-2}+y_{n-2}}{x_{n-2}+y_{n-1}}\right) \\
&\leq \frac{p}{q} + \left(\frac{\max(\delta_1,\delta_2,\epsilon_2)}{\min(D_2,E_1)}\right)\left(\frac{\max(B_2,C_1)}{\min(\beta_1,\beta_2,\gamma_2)}\right) x_n.
\end{aligned}$$

Let $b = \left(\frac{\max(\delta_1,\delta_2,\epsilon_2)}{\min(D_2,E_1)}\right)\left(\frac{\max(B_2,C_1)}{\min(\beta_1,\beta_2,\gamma_2)}\right)$. Using the above, we get

$$\begin{aligned}
y_n = \frac{p+\delta_1 x_{n-1}+\delta_2 x_{n-2}+\epsilon_2 y_{n-2}}{q+D_2 x_{n-2}+E_1 y_{n-1}} &\geq \frac{p+\delta_1 x_{n-1}+\delta_2 x_{n-2}}{q+D_2 x_{n-2}+E_1\left(\frac{p}{q}+b x_{n-1}\right)} \\
&\geq \frac{\min(p,\delta_1,\delta_2)}{\max(q+E_1\left(\frac{p}{q}\right),D_2,bE_1)}.
\end{aligned}$$

We now show that we may choose
$M_2 = \max\left(\frac{x_1}{y_1},\frac{x_2}{y_2},\frac{x_3}{y_3},\left(\frac{\max(1,\beta_1,\beta_2,\gamma_2)}{\min(\frac{C_1 L}{2},B_2,\frac{C_1}{2})}\right)\left(\frac{\max(q,D_2,E_1)}{\min(p,\delta_1,\delta_2,\epsilon_2)}\right)\right)$ since

$$\begin{aligned}
x_n = \frac{\beta_1 x_{n-1}+\beta_2 x_{n-2}+\gamma_2 y_{n-2}}{B_2 x_{n-2}+C_1 y_{n-1}} &\leq \frac{1+\beta_1 x_{n-1}+\beta_2 x_{n-2}+\gamma_2 y_{n-2}}{\frac{C_1 L}{2}+B_2 x_{n-2}+\frac{C_1}{2} y_{n-1}} \\
&\leq \left(\frac{\max(1,\beta_1,\beta_2,\gamma_2)}{\min(\frac{C_1 L}{2},B_2,\frac{C_1}{2})}\right)\left(\frac{1+x_{n-1}+x_{n-2}+y_{n-2}}{1+x_{n-2}+y_{n-1}}\right) \leq \\
&\left(\frac{\max(1,\beta_1,\beta_2,\gamma_2)}{\min(\frac{C_1 L}{2},B_2,\frac{C_1}{2})}\right)\left(\frac{\max(q,D_2,E_1)}{\min(p,\delta_1,\delta_2,\epsilon_2)}\right) y_n.
\end{aligned}$$

We may also choose $M_1 = \min\left(\frac{x_1}{y_1},\frac{x_2}{y_2},\frac{x_3}{y_3},\left(\frac{\min(\frac{\gamma_2 L}{2},\beta_1,\beta_2,\frac{\gamma_2}{2})}{\max(1,B_2,C_1)}\right)\left(\frac{\min(q,D_2,E_1)}{\max(p,\delta_1,\delta_2,\epsilon_2)}\right)\right)$ since

$$\begin{aligned}
x_n = \frac{\beta_1 x_{n-1}+\beta_2 x_{n-2}+\gamma_2 y_{n-2}}{B_2 x_{n-2}+C_1 y_{n-1}} &\geq \frac{\frac{\gamma_2 L}{2}+\beta_1 x_{n-1}+\beta_2 x_{n-2}+\frac{\gamma_2}{2} y_{n-2}}{1+B_2 x_{n-2}+C_1 y_{n-1}} \\
&\geq \left(\frac{\min(\frac{\gamma_2 L}{2},\beta_1,\beta_2,\frac{\gamma_2}{2})}{\max(1,B_2,C_1)}\right)\left(\frac{1+x_{n-1}+x_{n-2}+y_{n-2}}{1+x_{n-2}+y_{n-1}}\right) \\
&\geq \left(\frac{\min(\frac{\gamma_2 L}{2},\beta_1,\beta_2,\frac{\gamma_2}{2})}{\max(1,B_2,C_1)}\right)\left(\frac{\min(q,D_2,E_1)}{\max(p,\delta_1,\delta_2,\epsilon_2)}\right) y_n.
\end{aligned}$$

For the final condition needed to be satisfied, it is observed that $\{1,2\} = I_\beta \cup I_\gamma \subset I_B \cup I_C = \{1,2\}$.

$\square$

Example 5. *Consider the following system of two rational difference equations*

$$x_n = \frac{\gamma_1 y_{n-1} + \gamma_2 y_{n-2}}{A + B_2 x_{n-2} + C_2 y_{n-2}}, \quad n \in \mathbb{N},$$

$$y_n = \frac{p + \delta_1 x_{n-1} + \epsilon_1 y_{n-1} + \epsilon_2 y_{n-2}}{q + D_1 x_{n-1} + D_2 x_{n-2} + E_2 y_{n-2}}, \quad n \in \mathbb{N}.$$

We assume positive parameters and non-negative initial conditions. This implies that the solutions x_n and y_n are bounded above by a positive constant.

Proof. We apply Theorem 16. We first notice that there exists $M_1 > 0$ and $M_2 \geq 0$ so that $y_n \leq M_1 x_n + M_2$ for all $n \in \mathbb{N}$. This is since

$$y_n = \frac{p + \delta_1 x_{n-1} + \epsilon_1 y_{n-1} + \epsilon_2 y_{n-2}}{q + D_1 x_{n-1} + D_2 x_{n-2} + E_2 y_{n-2}} \leq \left(\frac{\max(p, \delta_1, \epsilon_1, \epsilon_2)}{\min(q, D_1, D_2, E_2)}\right)\left(\frac{1 + x_{n-1} + y_{n-1} + y_{n-2}}{1 + x_{n-1} + x_{n-2} + y_{n-2}}\right)$$

$$\leq \left(\frac{\max(p, \delta_1, \epsilon_1, \epsilon_2)}{\min(q, D_1, D_2, E_2)}\right)\left(\frac{1 + x_{n-1}}{1 + x_{n-1} + x_{n-2} + y_{n-2}}\right)$$

$$+ \left(\frac{\max(p, \delta_1, \epsilon_1, \epsilon_2)}{\min(q, D_1, D_2, E_2)}\right)\left(\frac{y_{n-1} + y_{n-2}}{1 + x_{n-1} + x_{n-2} + y_{n-2}}\right)$$

$$\leq \left(\frac{\max(p, \delta_1, \epsilon_1, \epsilon_2)}{\min(q, D_1, D_2, E_2)}\right) + \left(\frac{\max(p, \delta_1, \epsilon_1, \epsilon_2)}{\min(q, D_1, D_2, E_2)}\right)\left(\frac{y_{n-1} + y_{n-2}}{1 + x_{n-2} + y_{n-2}}\right)$$

$$\leq M_1 x_n + M_2$$

where $M_1 = \left(\frac{\max(p,\delta_1,\epsilon_1,\epsilon_2)}{\min(q,D_1,D_2,E_2)}\right)\left(\frac{\max(A,B_2,C_2)}{\min(\gamma_1,\gamma_2)}\right)$ and $M_2 = \left(\frac{\max(p,\delta_1,\epsilon_1,\epsilon_2)}{\min(q,D_1,D_2,E_2)}\right)$. For the final condition, let $\eta = 2$ so that for any sequence $\{c_m\}_{m=1}^{\infty}$ with $c_m \in I_\beta \cup (I_\gamma \backslash I_C) = \{1\}$ for $m = 1, 2, \ldots$ we choose $N_1 = 1$,$N_2 = 2 \leq \eta$, such that $\sum_{m=N_1}^{N_2} 1 \in I_B = \{2\}$. $\square$

Example 6. *Consider the following first order system of two rational difference equations*

$$x_n = \frac{\alpha + \gamma_1 y_{n-1}}{A + B_1 x_{n-1}}, \quad n \in \mathbb{N},$$

$$y_n = \frac{\delta_1 x_{n-1} + \epsilon_1 y_{n-1}}{q + D_1 x_{n-1}}, \quad n \in \mathbb{N}.$$

We assume positive parameters and non-negative initial conditions. This implies that the solutions x_n and y_n are bounded above by a positive constant.

Proof. We apply Theorem 20. First notice that, by Theorem 27, there exists $M_1, M_3 > 0$ and $M_4 \geq M_2 \geq 0$ so that $x_n \leq M_1 y_n + M_2 \leq M_3 x_n + M_4$ for all $n \in \mathbb{N}$. This is since

$$\begin{gathered}\{1\} = I_\delta \subset I_\beta \cup I_B = \{1\}\\ \emptyset = I_\beta \subset I_\delta \cup I_D\\ \{1\} = I_B = I_D\\ \{1\} = I_\epsilon \subset I_\gamma \cup I_C = \{1\}\\ \{1\} = I_\gamma \subset I_\epsilon \cup I_E = \{1\}\\ \emptyset = I_C = I_E\\ \alpha > 0 \quad \textit{and} \quad A > 0 \quad \textit{with} \quad q > 0.\end{gathered}$$

For the final condition, let $\eta = 1$ so that for any sequence $\{c_m\}_{m=1}^{\infty}$ with $c_m \in I_\beta \cup I_\gamma = \{1\}$ for $m = 1, 2, \ldots$ we choose $N_1 = 1$, $N_2 = 1 \leq \eta$ so that $\sum_{m=N_1}^{N_2} 1 \in I_B \cup I_C = \{1\}$. ☐

Example 7. *Consider the following first order system of two rational difference equations*

$$x_n = \frac{\beta_1 x_{n-1} + \gamma_1 y_{n-1}}{A + B_1 x_{n-1}}, \quad n \in \mathbb{N},$$

$$y_n = \frac{p + \delta_1 x_{n-1} + \epsilon_1 y_{n-1}}{q + D_1 x_{n-1}}, \quad n \in \mathbb{N}.$$

We assume positive parameters and non-negative initial conditions. This implies that the solutions x_n and y_n are bounded above by a positive constant.

Proof. We apply Theorem 21. First notice that, by Theorem 27, there exists $M_1, M_3 > 0$ and $M_4 \geq M_2 \geq 0$ so that $x_n \leq M_1 y_n + M_2 \leq M_3 x_n + M_4$ for all $n \in \mathbb{N}$. This is since

$$\begin{gathered}\{1\} = I_\delta \subset I_\beta \cup I_B = \{1\}\\ \{1\} = I_\beta \subset I_\delta \cup I_D = \{1\}\\ \{1\} = I_B = I_D\\ \{1\} = I_\epsilon \subset I_\gamma \cup I_C = \{1\}\\ \{1\} = I_\gamma \subset I_\epsilon \cup I_E = \{1\}\\ \emptyset = I_C = I_E\\ p > 0 \quad \textit{and} \quad q > 0 \quad \textit{with} \quad A > 0.\end{gathered}$$

For the final condition, let $\eta = 1$ so that for any sequence $\{c_m\}_{m=1}^{\infty}$ with $c_m \in I_\delta \cup I_\epsilon = \{1\}$ for $m = 1, 2, \ldots$ we choose $N_1 = 1$, $N_2 = 1 \leq \eta$ so that $\sum_{m=N_1}^{N_2} 1 \in I_D \cup I_E = \{1\}$. ☐

Example 8. *Consider the following first order system of two rational difference equations*

$$x_n = \frac{\alpha + \beta_1 x_{n-1}}{B_1 x_{n-1} + C_1 y_{n-1}}, \quad n \in \mathbb{N},$$

$$y_n = \frac{p + \epsilon_1 y_{n-1}}{D_1 x_{n-1} + E_1 y_{n-1}}, \quad n \in \mathbb{N}.$$

We assume positive parameters and non-negative initial conditions. This implies that the solutions x_n and y_n are bounded above by a positive constant.

Proof. We apply Theorem 22 case (i). First notice that, by Theorem 27, there exists $M_1, M_3 > 0$ and $M_4 \geq M_2 \geq 0$ so that $x_n \leq M_1 y_n + M_2 \leq M_3 x_n + M_4$ for all $n \in \mathbb{N}$. This is since

$$\begin{gathered} \emptyset = I_\delta \subset I_\beta \cup I_B \\ \{1\} = I_\beta \subset I_\delta \cup I_D = \{1\} \\ \{1\} = I_B = I_D \\ \{1\} = I_\epsilon \subset I_\gamma \cup I_C = \{1\} \\ \emptyset = I_\gamma \subset I_\epsilon \cup I_E \\ \{1\} = I_C = I_E \\ \alpha > 0 \quad and \quad p > 0. \end{gathered}$$

Thus $x_n \leq M_1 y_n + M_2 + 1 \leq M_3 x_n + M_4 + 2$ for all $n \in \mathbb{N}$.

Case (i) is satisfied, since $\{1\} = I_B \cup I_C \subset I_\beta \cup I_\gamma = \{1\}$, $\alpha > 0$, and $I_B \neq \emptyset$. For the final condition, let $\eta = 1$ so that for any sequence $\{c_m\}_{m=1}^{\infty}$ with $c_m \in I_\beta \cup I_\gamma = \{1\}$ for $m = 1, 2, \ldots$ we choose $N_1 = 1$, $N_2 = 1 \leq \eta$ so that $\sum_{m=N_1}^{N_2} 1 \in I_B \cup I_C = \{1\}$. ☐

Example 9. *Consider the following system of two rational difference equations*

$$x_n = \frac{\alpha + \beta_2 x_{n-2} + \gamma_1 y_{n-1} + \gamma_2 y_{n-2}}{A + B_2 x_{n-2} + C_1 y_{n-1} + C_2 y_{n-2}}, \quad n \in \mathbb{N},$$

$$y_n = \frac{p + \delta_2 x_{n-2} + \epsilon_1 y_{n-1} + \epsilon_2 y_{n-2}}{D_2 x_{n-2} + E_2 y_{n-2}}, \quad n \in \mathbb{N}.$$

We assume positive parameters and non-negative initial conditions. This implies that the solutions x_n and y_n are bounded above by a positive constant.

Proof. We apply Theorem 14 case (ii). To apply Theorem 14 we must use a change of variables. The change of variables we refer to here comes from renaming x_n as y_n, β_i as ϵ_i, B_i as E_i, γ_i as δ_i, C_i as D_i, α as p, A as q, and vice versa.
Although we use this change of variables, we keep notation consistent with our notation before the change of variables for the remainder of this example. We do this to avoid confusion. First notice that, by applying Theorem 24 after the change of variables, we see that there exists $M_1 > 0$ so that $x_n \leq M_1 y_n$ for all $n \in \mathbb{N}$. This is since

$$\begin{gathered} \{1,2\} = I_\gamma = I_\epsilon \\ \{2\} = I_E \subset I_C = \{1,2\} \\ \{2\} = I_\beta = I_\delta \\ \{2\} = I_D = I_B = \{2\} \\ \alpha > 0 \quad and \quad p > 0. \end{gathered}$$

The conditions for case (ii) are satisfied since

$$\begin{gathered} \alpha > 0 \\ \{1,2\} = I_C = I_\gamma \\ \{2\} = I_B = I_\beta \\ I_D \neq \emptyset. \end{gathered}$$

For the final condition let $\eta_1 = 1$ so that for any sequence $\{c_m\}_{m=1}^{\infty}$ with $c_m \in I_\beta = \{2\}$ for $m = 1, 2, \ldots$ we choose $N_1 = 1, N_2 = 1 \leq \eta_1$, since $c_1 = 2$, so that $\sum_{m=N_1}^{N_2} 2 \in$

$I_C \cup I_B = \{1,2\}$. Now, we let $\eta_2 = 2$ so that for any sequence $\{d_m\}_{m=1}^{\infty}$ with $d_m \in I_\epsilon = \{1,2\}$, for $m = 1,2,\ldots$, we choose $N_3 = 1, N_4 = 1 \leq \eta_2$, if $d_1 = 2$, so that $\sum_{m=N_3}^{N_4} d_m = 2 \in I_E = \{2\}$, we choose $N_3 = 1 \leq \eta_2$ and $N_4 = 2 \leq \eta_2$, if $d_1 = 1$ and $d_2 = 1$, so that $\sum_{m=N_3}^{N_4} d_m = 2 \in I_E = \{2\}$ and we choose $N_3 = 2, N_4 = 2 \leq \eta_2 = 2$, if $d_2 = 2$ with $d_1 = 1$, so that $\sum_{m=N_3}^{N_4} d_m \in I_E = \{2\}$. □

Example 10. *Consider the following system of two rational difference equations*

$$x_n = \frac{\beta_1 x_{n-1} + \beta_2 x_{n-2}}{B_2 x_{n-2} + C_1 y_{n-1}}, n = 0,1,2,\ldots,$$

$$y_n = \frac{p + \delta_1 x_{n-1} + \delta_2 x_{n-2}}{q + D_2 x_{n-2} + E_1 y_{n-1}}, n = 0,1,2,\ldots.$$

We assume positive parameters and positive initial conditions. This implies that the solutions x_n and y_n are bounded above by a positive constant.

Proof. We apply Theorem 22 case (ii). We will now prove that there exists $M_1, M_3 > 0$ and $M_4 > M_2 > 0$ so that $x_n \leq M_1 y_n + M_2 \leq M_3 x_n + M_4$ for all $n \in \mathbb{N}$. We now show that we may choose $M_3 = \max\left(M_1\left(\frac{\max(\delta_1,\delta_2)}{\min(D_2,E_1)}\right)\left(\frac{\max(B_2,C_1)}{\min(\beta_1,\beta_2)}\right), \frac{M_1 y_1}{x_1}\right)$ and $M_4 = M_1(\frac{p}{q}+1) + M_2$. This is since

$$y_n = \frac{p + \delta_1 x_{n-1} + \delta_2 x_{n-2}}{q + D_2 x_{n-2} + E_1 y_{n-1}} \leq \frac{p}{q} + \frac{\delta_1 x_{n-1} + \delta_2 x_{n-2}}{D_2 x_{n-2} + E_1 y_{n-1}} \leq$$

$$\frac{p}{q} + \left(\frac{\max(\delta_1,\delta_2)}{\min(D_2,E_1)}\right)\left(\frac{x_{n-1}+x_{n-2}}{x_{n-2}+y_{n-1}}\right) \leq \frac{p}{q} + \left(\frac{\max(\delta_1,\delta_2)}{\min(D_2,E_1)}\right)\left(\frac{\max(B_2,C_1)}{\min(\beta_1,\beta_2)}\right) x_n.$$

Let $b = \left(\frac{\max(\delta_1,\delta_2)}{\min(D_2,E_1)}\right)\left(\frac{\max(B_2,C_1)}{\min(\beta_1,\beta_2)}\right)$. To deduce M_1 and M_2 we show that there exists L so that $y_n \geq L$ for all $n \in \mathbb{N}$. So,

$$y_n = \frac{p + \delta_1 x_{n-1} + \delta_2 x_{n-2}}{q + D_2 x_{n-2} + E_1 y_{n-1}} \geq \frac{p + \delta_1 x_{n-1} + \delta_2 x_{n-2}}{q + D_2 x_{n-2} + E_1\left(\frac{p}{q} + b x_{n-1}\right)} \geq \frac{\min(p,\delta_1,\delta_2)}{\max(q + E_1\frac{p}{q}, D_2, bE_1)}.$$

Hence, $L = \min\left(\frac{\min(p,\delta_1,\delta_2)}{\max(q+E_1\frac{p}{q},D_2,bE_1)}, y_1\right)$. Now we show that we may choose $M_1 = \max\left(\left(\frac{\max(1,\beta_1,\beta_2)}{\min(\frac{C_1 L}{2},B_2,\frac{C_1}{2})}\right)\left(\frac{\max(q,D_2,E_1)}{\min(p,\delta_1,\delta_2)}\right), \frac{x_n}{y_n}\right)$ and $M_2 = 1$. So,

$$x_n = \frac{\beta_1 x_{n-1} + \beta_2 x_{n-2}}{B_2 x_{n-2} + C_1 y_{n-1}} \leq \frac{1 + \beta_1 x_{n-1} + \beta_2 x_{n-2}}{\frac{C_1 L}{2} + B_2 x_{n-2} + \frac{C_1}{2} y_{n-1}}$$

$$\leq \left(\frac{\max(1,\beta_1,\beta_2)}{\min(\frac{C_1 L}{2}, B_2, \frac{C_1}{2})}\right)\left(\frac{1 + x_{n-1} + x_{n-2}}{1 + x_{n-2} + y_{n-1}}\right) \leq$$

$$\left(\frac{\max(1,\beta_1,\beta_2)}{\min(\frac{C_1 L}{2}, B_2, \frac{C_1}{2})}\right)\left(\frac{\max(q,D_2,E_1)}{\min(p,\delta_1,\delta_2)}\right) y_n + 1.$$

The conditions of case (ii) are satisfied since $\{1,2\} = I_D \cup I_E = I_\delta \cup I_\epsilon = \{1,2\}$, $p > 0$ and $I_C \neq \emptyset$. For the final condition, let $\eta = 1$ so that for any sequence $\{c_m\}_{m=1}^{\infty}$

with $c_m \in I_\beta \cup I_\gamma = \{1,2\}$ for $m = 1,2,\ldots$ we choose $N_1 = 1$, $N_2 = 1 \leq \eta$ so that $\sum_{m=N_1}^{N_2} c_m = c_1 \in I_B \cup I_C = \{1,2\}$, since $c_1 \in \{1,2\}$.

□

8. Conclusion

We have presented numerous techniques which apply the method of iteration to systems of rational difference equations. These techniques provide a starting point for the immense task of understanding the boundedness character of systems of rational difference equations of order greater than one. There are three directions for further work which we feel have promise. One important goal is to provide some type of comprehensive criterion which, when satisfied, guarantees the success of the boundedness by iteration technique. Theorem 6 in [4] provided this type of criterion for rational difference equations of order greater than one. We feel that a similar approach is required for systems of rational difference equations of order greater than one. Another important direction for further work is to apply similar techniques to those presented above for systems of three or more rational difference equations. Analogues to the ideas in [5] for systems of rational difference equations would be another direction of interest. See [1], [2], and [6] for further work on systems of rational difference equations.

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